Exam 1 Solutions

1. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   Circle yes or no to the following questions. You do not need to show work on this problem. (12 pts)

   (a) Is A a diagonal matrix? Yes

   (b) Is A a scalar matrix? No

   (c) Is A an upper triangular matrix? Yes

   (d) Is A a symmetric matrix? Yes

   (e) Is A an invertible matrix? No

   (f) Is A in row echelon form? No

2. The following 2 questions are multiple choice. Circle one answer for each question. You do not need to show work on this problem. (8 pts)

   (a) Suppose A is a 5×5 matrix and $\det(A) = 3$. What can you say about the reduced row echelon form (RREF) of A?

   (iii) It is I_5

   $\det(A) \neq 0$ so A is invertible and the RREF of A is the identity of the same size as A which is I_5.

   (b) Suppose $\mathbf{v_1}$ and $\mathbf{v_2}$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \neq \mathbf{0}$. What linear system is $\mathbf{v_1} + 2\mathbf{v_2}$ a solution to?

   (i) $A\mathbf{x} = 3\mathbf{b}$

   $A(\mathbf{v_1} + 2\mathbf{v_2}) = A\mathbf{v_1} + 2A\mathbf{v_2} = \mathbf{b} + 2\mathbf{b} = 3\mathbf{b}$.

3. Is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ a linear combination of the vectors

   $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$? Why or why not? (12 pts)

   To see if $\mathbf{v}$ is a linear combination of the other three vectors, we need to see if we can find x, y, z such that

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x+3y-2z \\ 2x+y+z \\ y-z \end{bmatrix}$. This is equivalent to checking if the linear system $x + 3y - 2z = 1, 2x + y + z = 1, y - z = 1$ has any solutions. This linear system has augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{array}\right]$.

The row operations $r_2 - 2r_1 \to r_2, r_2 + 5r_3 \to r_2$ take this to $\left[\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right]$.

The second row is the equation $0 = 4$ so it has no solutions and $\mathbf{v}$ is not a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

4. Suppose A and B are invertible matrices with $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & 1 \end{bmatrix}$ and

$$B^{-1} = \begin{bmatrix} 1 & -4 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

(a) Compute $(BA^T)^{-1}$. (12 pts)

$(BA^T)^{-1} = (A^T)^{-1}B^{-1} = (A^{-1})^T B^{-1} =$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & -3 & 3 \\ 1 & -3 & 1 \end{bmatrix}$$

(b) Let $\mathbf{c}$ be the vector $\mathbf{c} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$. Find all solutions to the linear system $BA^T\mathbf{x} = \mathbf{c}$. (6 pts)

The solutions are $\mathbf{x} = (BA^T)^{-1}\mathbf{c} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & -3 & 3 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \\ 1 \end{bmatrix}$

5. Let $A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$.

(a) Compute $\det(A)$. (12 pts)

There are lots of ways to compute this determinant. One way to do this would be to use cofactor expansion. If you expand along row 2 twice, you get $\det(A) = -5\det\left(\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\right) = (-5)(-1)\det\left(\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\right) =$ $(-5)(-1)(1-6) = -25.$

Another way would be using reduction to triangular form. The row operations $r_1 \leftrightarrow r_2, r_4 - 2r_2 \to r_4$ give you the upper triangular matrix $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$. This matrix has determinant 25 (the product of the diagonals). The row operations we did were type 1 and 3. The type 3 doesn't change the determinant but the type 1 changed the sign, so $\det(A) = -25$.

This matrix can also be done using the definition of the determinant - there would be 2 nonzero terms. Or it can be done with a combination of different methods, like cofactor expansion to get it down to 3×3 determinants then the 3×3 trick to find those determinants.

(b) One of the following matrices is A^{-1}, which one is it? Circle your answer. (5 pts)

(iii) $\begin{bmatrix} 0 & \frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & 0 & 0 & \frac{3}{5} \\ 0 & 0 & -1 & 0 \\ \frac{2}{5} & 0 & 0 & -\frac{1}{5} \end{bmatrix}$

The easiest way to do this is probably just to pick one and start multiplying by A. If at any point you get an entry that doesn't match I_4, then try another one. You can also use the method where you look at the matrix $[A : I_4]$ and do row operations to get to $[I_4 : A^{-1}]$, but this probably takes longer.

6. Find all solutions to the following linear system using Gaussian elimination or Gauss-Jordan reduction. Make sure to specify the row operations that you are doing and write your answer as a vector. (16 pts)

$$\begin{aligned} x + y - z - 3w &= 6 \\ y + 2z &= 4 \\ x + y - z - 2w &= 7 \\ 2x + y - 4z - 6w &= 8 \end{aligned}$$

The augmented matrix of this linear system is $\left[\begin{array}{cccc|c} 1 & 1 & -1 & -3 & 6 \\ 0 & 1 & 2 & 0 & 4 \\ 1 & 1 & -1 & -2 & 7 \\ 2 & 1 & -4 & -6 & 8 \end{array}\right]$. The row operations $r_3 - r_1 \to r_3, r_4 - 2r_1 \to r_4, r_4 + r_2 \to r_4$ give you the matrix $\left[\begin{array}{cccc|c} 1 & 1 & -1 & -3 & 6 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. This is in REF and you can either use back substitution to solve from here, or put the matrix in RREF. The row operations $r_1 + 3r_3 \to r_1, r_1 - r_2 \to r_1$ will give you the RREF which is $\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The variable z can be anything since the column corresponding to z does not contain a leading one. The columns corresponding to the other variables have leading ones so they can be solved for in terms of z. The equations from RREF are $x - 3z = 5, y + 2z = 4, w = 1$ so the solutions are all vectors of the form $\begin{bmatrix} 5+3z \\ 4-2z \\ z \\ 1 \end{bmatrix}$ where z can be anything.

7. Let A be an $n \times n$ skew symmetric matrix.

   (a) Show that if n is odd, then A is not invertible. (12 pts)
   Hint: Use determinants and their properties.

   A skew symmetric so $A^T = -A$. Taking determinants, this means $\det(A^T) = \det(-A)$. Using properties of determinants, $\det(A^T) = \det(A)$ and $\det(-A) = (-1)^n \det(A) = -\det(A)$, so $\det(A) = -\det(A)$. Note that we used the fact that n was odd to get that $(-1)^n = -1$. The equation $\det(A) = -\det(A)$ means that $\det(A) = 0$ so A is not invertible.

   (b) Show that part (a) is not true for n even by finding an invertible 2×2 skew symmetric matrix. (5 pts)

   The 2×2 skew symmetric matrices look like $\begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}$ for a constant k. The determinant of a matrix of this form is k^2 so as long as $k \neq 0$, the matrix will be invertible. Hence any matrix of the form $\begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix}$ with $k \neq 0$ would be an example of a skew symmetric invertible matrix.