MATH 2443
3rd Midterm Review Solutions

1. Evaluate $\int_C xz\ dx - z\ dy + y\ dz$ where C is the line segment from $(1,1,1)$ to $(2,3,4)$.

   The line segment can be parametrized as $x = 1 + t, y = 1 + 2t, z = 1 + 3t$ where $0 \le t \le 1$. Then $dx = dt, dy = 2dt, dz = 3dt$ so the integral becomes $\int_0^1 (1+t)(1+3t)\ dt - (1+3t)\ 2dt + (1+2t)\ 3dt = \int_0^1 3t^2 + 4t + 2\ dt =$ $t^3 + 2t^2 + 2t\Big|_0^1 = 5$.

2. The force field $F(x,y) = \langle e^{x^2}, 2x - e^{y^2} \rangle$ acts on a particle moving from $(0,0)$ to $(1,1)$.

   (a) Compute the work done by the force if the particle moves in a straight line.

   The work is $\int_C e^{x^2}\ dx + (2x - e^{y^2})\ dy$. The line can be parametrized as $x = t, y = t$ for $0 \le t \le 1$ so both dx, dy equal dt and the integral becomes $\int_0^1 e^{t^2} + 2t - e^{t^2}\ dt = \int_0^1 2t\ dt = t^2\Big|_0^1 = 1$.

   (b) Compute the work done by the force if the particle moves first along the x-axis to $(4,0)$ and then in a straight line to $(1,1)$.

   These are not easy to evaluate directly, so we will instead use Green's theorem and the results from part a. Let C be the triangle with vertices $(0,0), (4,0), (1,1)$ oriented counterclockwise and T the region enclosed by C. Let C_1 be the path from part b, a straight line from $(0,0)$ to $(4,0)$ then a straight line to $(1,1)$. Let C_2 be the line from $(1,1)$ to $(0,0)$. Then $C = C_1 \cup C_2$ so

   $$\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr\ .$$

   Then $\int_{C_1} F \cdot dr = -1$ because C_2 is the negative of the curve in part a. We can calculate $\int_C F \cdot dr$ using Green's theorem. We have that $P = e^{x^2}, Q = 2x - e^{y^2}$ so $Q_x = 2, P_y = 0$ and by Green's theorem, $\int_C F \cdot dr = \iint_T 2\ dA = 2A(T) = 4$ where $A(T)$ is the area of the triangle T. Then $4 = -1 + \int_{C_2} F \cdot dr$ so the work over C_2 is $\int_{C_2} F \cdot dr = 5$.

3. Evaluate $\int_C (1 + yz)\ dx + (2y + xz)\ dy + (-3x^2)\ dz$ along the path parametrized by $x = t, y = t^2, z = e^t$ for $0 \le t \le 1$.

   We have that $dx = dt,\ dy = 2tdt,\ dz = e^t dt$ so this becomes $\int_0^1 (1 + t^2 e^t) + (2t^2 + te^t)2t + (-3t^2)e^t\ dt = \int_0^1 1 + 4t^3\ dt = t + t^4\Big|_0^1 = 2$.

4. Find the mass of that part of the surface $z = xy$ that lies within one unit of the z-axis if the density at the point (x, y) is given by $\delta(x, y) = x^2 + y^2$. Note: The mass of an object is equal to the integral over the object of the density function.

   This is the surface integral of the function $x^2 + y^2$ over the part of the surface $z = xy$ which is inside the cylinder $x^2 + y^2 = 1$. If D is the unit disk $x^2 + y^2 \leq 1$ on the xy-plane then this integral is $\iint_D (x^2 + y^2)\ dS$. Let $f(x, y) = xy$. Then $dS = \sqrt{1 + (f_x)^2 + (f_y)^2} dA = \sqrt{1 + y^2 + x^2} dA$ so the integral is $\iint_D (x^2 + y^2)\sqrt{1 + y^2 + x^2}\ dA$. Changing to polar, we get the integral

$$\int_0^{2\pi} \int_0^1 r^2\sqrt{1 + r^2} r\ dr\ d\theta = \int_0^{2\pi} \int_0^1 r^3\sqrt{1 + r^2}\ dr\ d\theta = 2\pi \int_0^1 r^3\sqrt{1 + r^2}\ dr\ .$$

   Use the u-substitution $u = 1 + r^2$, $du = 2rdr$ and $r^2 = u - 1$ to get that this is

$$\pi \int_1^2 (u-1)\sqrt{u}\, du = \pi \int_1^2 u^{\frac{3}{2}} - u^{\frac{1}{2}}\ du = \pi \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^2 = 2\pi\left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \frac{1}{5} + \frac{1}{3}\right).$$

5. A nonuniform piece of wire if bent into the shape of the curve $y = \sin(x)$ between $x = 0$ and $x = \pi$. The density of the wire at the point (x, y) is equal to $1 + y$. Set up, but do not evaluate, an integral equal to the mass of the wire.

   This is the line integral of $1 + y$ over the curve C parametrized by $x = t, y = \sin(t), 0 \leq t \leq \pi$ so it's $\int_C (1 + y)\ ds$. Then $dx/dt = 1, dy/dt = \cos(t)$ so $ds = \sqrt{1 + \cos^2(t)} dt$ and the integral is

$$\int_0^{\pi} (1 + \sin(t))\sqrt{1 + \cos^2(t)}\ dt\ .$$

6. Find the work done by the force $F(x, y) = \langle 2x\cos(x^2) + e^y, xe^y\rangle$ in moving a particle along a semicircle of radius 1 from $(1, 0)$ to $(-1, 0)$.

   This would be difficult to evaluate directly so we will check if F is conservative so we can use the fundamental theorem of line integrals. F is defined on the entire xy-plane and $P_y = e^y = Q_x$ so F is conservative. F has potential function $f(x, y) = \sin(x^2) + xe^y$ so the work is $f(-1, 0) - f(1, 0) = (\sin(1) - 1) - (\sin(1) + 1) = -2$.

7. A force given by $F(x, y) = \langle y, e^x\rangle$ acts on a particle moving from the point $(0, 1)$ to the point $(2, 0)$ along the following path: first along the curve $y = e^x$ from $(0, 1)$ to $(2, e^2)$ and then along a line from there to $(2, 0)$. Find the work done by the force.

Let C_1 be the first curve. Then C_1 can be parametrized by $x = t, y = e^t$, $0 \leq 2$ and $dx = dt, dy = e^t dt$ so the work is
$\int_{C_1} y\ dx + e^x\ dy = \int_0^2 e^t + e^{2t}\ dt = e^t + \frac{1}{2}e^{2t}\Big|_0^2 = e^2 + \frac{1}{2}e^4 - \frac{3}{2}$.

Let C_2 be the second part of the curve, the line from $(2, e^2)$ to $(2, 0)$. This can be parametrized by $x = 2, y = -t$, $-e^2 \leq t \leq 0$. Then $dx = 0, dy = -dt$ and the work is $\int_{C_2} y\ dx + e^x\ dy = \int_{-e^2}^{0} -e^2\ dt = -e^2 t\Big|_{-e^2}^{0} = -e^4$.

The work over the whole path is $e^2 + \frac{1}{2}e^4 - \frac{3}{2} - e^4 = -\frac{1}{2}e^4 + e^2 - \frac{3}{2}$.

8. Evaluate $\int_C 2xe^y\ dx + (3x + x^2 e^y)\ dy$ where C is the triangular path from $(0, 0)$ to $(1, 1)$ to $(2, 0)$ and back to $(0, 0)$.

   Let T be the triangular region enclosed by C. Then by Green's theorem, the line integral over the triangle traversed counterclockwise is
   $\iint_T 3 + 2xe^y - 2xe^y\ dA = \iint_T 3\ dA = 3A(T) = 3$ where $A(T) = 1$ is the area of the triangle. The path C traverses the triangle clockwise, so the line integral over C will be the negative of the counterclockwise integral so it is -3.

9. Let $F(x, y, z) = \langle e^y + ze^x, xe^y - e^z, e^x - ye^z \rangle$.

   (a) Compute the curl and divergence of F.

   The curl of F is $\langle -e^z - (-e^z), e^x - e^x, e^y - e^y \rangle = \langle 0, 0, 0 \rangle$ and the divergence is $ze^x + xe^y - ye^z$.

   (b) Determine if F is conservative. If yes, find f such that $F = \nabla f$, if not explain why.

   F is conservative as it is defined on all $\mathbb{R}^3$ and has curl equal to 0. To find f, we integrate the first component of F with respect to x and get that $f(x, y, z) = xe^y + ze^x + g(y, z)$. Then $xe^y - e^z = f_y = xe^y + g_y(y, z)$ so $g_y(y, z) = -e^z$. Integrating with respect to y we get $g(y, z) = -ye^z + h(z)$ so $f(x, y, z) = xe^y + ze^x - ye^z + h(z)$. Then $e^x - ye^z = f_z = e^x - ye^z + h'(z)$ so $h'(z) = 0$ and $h(z) = c$ where c is constant. Thus f is of the form $f(x, y, z) = xe^y + ze^x - ye^z + c$. We only need to find one such f so in particular we can take the one where $c = 0$ so we have $f(x, y, z) = xe^y + ze^x - ye^z$.

   (c) Integrate $\int_C F \cdot dr$ along the line segment from $(1, 1, 1)$ to $(2, 2, 2)$.

   F is conservative so we can use the fundamental theorem of line integrals to get that
   $\int_C F \cdot dr = f(2, 2, 2) - f(1, 1, 1) = (2e^2 + 2e^2 - 2e^2) - (e + e - e) = 2e^2 - e$.

10. Compute $\int \cos(y^2)dx + x(x - 2y\sin(y^2))dy$ along each of the following paths.

   (a) The line segment from $(0, 0)$ to $(1, 0)$.

Parameterize this as $x = t, y = 0$ with $0 \le t \le 1$ and $dx = dt, dy = 0$. Then the integral becomes $\int_0^1 \, dt = 1$.

(b) The line segment from $(0, 1)$ to $(0, 0)$.

Parameterize this as $x = 0, y = -t$ with $-1 \le t \le 0$. Then $dx = 0, dy = -dt$ and the integral becomes $\int_{-1}^0 0 \, dt = 0$.

(c) The line segment from $(0, 1)$ to $(1, 0)$.

This integral is hard to compute directly so instead use Green's theorem and the results from the other two parts. Let C be the triangle from $(0, 0)$ to $(1, 0)$ to $(0, 1)$ and back to $(0, 0)$ and T the region enclosed by C. Let C_1, C_2, C_3 be the three line segments in parts a,b,c respectively. Then $C = C_1 \cup -C_3 \cup C_2$ so $\int_C F \cdot dr = \int_{C_1} F \cdot dr - \int_{C_3} F \cdot dr + \int_{C_2} F \cdot dr$. By Green's theorem, $\int_C F \cdot dr = \iint_T 2x - 2y\sin(y^2) - (-2y\sin(y^2)) \, dA = \iint_T 2x \, dA = \int_0^1 \int_0^{1-x} 2x \, dydx = \int_0^1 2x(1-x) \, dx = \int_0^1 2x - 2x^2 \, dx = x^2 - (2/3)x^3\Big|_0^1 = 1 - (2/3)$. We combine this result with the results of parts a,b to get $1 - (2/3) = 1 - \int_{C_3} F \cdot dr + 0$ so the line integral over C_3 is $2/3$.

11. Let F be the force field $F(x, y) = \langle x, \sqrt{x^2 + y^2}\rangle$. A particle the feels this force starts at the origin. It is moved along the x-axis to the point $(1, 0)$ and then it is moved along a quarter circle centered at the origin until it reaches the point $(0, 1)$. Finally the particle is returned to the origin along the y-axis. Compute the total work done by the force field on the particle during this round trip.

Let D be the quarter D enclosed by the path the particle moves around. Then by Green's theorem, the work is

$$\iint_D \frac{x}{\sqrt{x^2 + y^2}} \, dA = \int_0^{\pi/2} \int_0^1 r\cos(\theta) \, drd\theta = \int_0^{\pi/2} \frac{1}{2}r^2 \cos(\theta)\Big|_0^1$$

$$= \int_0^{\pi/2} \frac{1}{2}\cos(\theta) \, d\theta = \frac{1}{2}\sin(\theta)\Big|_0^{\pi/2} = \frac{1}{2} \, .$$

12. A certain force $F = \langle P, Q, R\rangle$ is not completely known. It is known however that $P = yze^{xy} - y^2 + Axz$, $Q = xze^{xy} + 2e^z + Bxy$, and $R = e^{xy} + 3x^2 + Cye^z$ where A, B, C are constants. A particle is moved from $(0, 0, 0)$ to $(1, 1, 1)$ many times along different paths and it is found that the work done by the force is the same each time. Determine values of A, B, C which might explain this result. Compute the work done by the force using these values of A, B, C.

This result implies that F is probably conservative. We need to find values of A, B, C which make F conservative. We need $P_y = Q_x$ so $xyze^{xy} + ze^{xy} - 2y = xyze^{xy} + ze^{xy} + By$ and it follows that $B = -2$. We also need $P_z = R_x$ so $ye^{xy} + Ax = ye^{xy} + 6x$ so $A = 6$. Finally we need $Q_z = R_y$ so $xe^{xy} + 2e^z = xe^{xy} + Ce^z$ so $C = 2$. So F is conservative (and thus the line integral is independent of path) if $A = 6, B = -2, C = 2$.

To find the work we need to find a potential function f for F. Integrating P with respect to x we get that $f(x, y, z) = ze^{xy} - xy^2 + 3x^2 z + g(y, z)$. The derivative is $xze^{xy} + 2e^z - 2xy = f_y = xze^{xy} - 2xy + g_y(y, z)$ so $g_y(y, z) = 2e^z$. Integrating with respect to y we get that $g(y, z) = 2ye^z + h(z)$ and so $f(x, y, z) = ze^{xy} - xy^2 + 3x^2 z + 2ye^z + h(z)$. Then $e^{xy} + 3x^2 + 2ye^z = f_z = e^{xy} + 3x^2 + 2ye^z + h'(z)$ so $0 = h'(z)$ and we can take $h(z) = 0$ and $f(x, y, z) = ze^{xy} - xy^2 + 3x^2 z + 2ye^z$. Then the work done is $f(1, 1, 1) - f(0, 0, 0) = (e - 1 + 3 + 2e) - 0 = 3e + 2$.

13. A thin hollow shell has the shape of the paraboloid $z = 9 - x^2 - y^2$ for $z \geq 0$. Find the surface area of the shell.

    The (x, y) values with make $z \geq 0$ are $9 - x^2 - y^2 \geq 0$ which is $9 \geq x^2 + y^2$, a disk of radius 3 centered at the origin. Let D be this disk and $f(x, y) = 9 - x^2 - y^2$. Then the surface area is $\iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \iint_D \sqrt{1 + (-2x)^2 + (-2y^2)}\, dA = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$ Changing to polar we get $\int_0^{2\pi} \int_0^3 r\sqrt{1 + 4r^2}\, dr d\theta = 2\pi \int_0^3 r\sqrt{1 + 4r^2}\, dr$. Using u-substitution with $u = 1 + 4r^2$ we get $2\pi \int_1^{37} (1/8) u^{1/2}\, du = (2\pi)(1/8)(2/3) u^{3/2}\Big|_1^{37} = \frac{\pi}{6}(37\sqrt{37} - 1)$.

14. A sphere of radius 2 is centered at the origin. Find the area of that part of the sphere that lies above the region on the (x, y)-plane where $x \geq 0, y \geq 0$ and $x^2 + y^2 \leq 1$.

    Let D be the quarter disk $x^2 + y^2 \leq 1$, $x, y \geq 0$. The formula for the sphere is $x^2 + y^2 + z^2 = 4$ and we're looking at part of the sphere above the (x, y)-plane so we're looking at the top part of the sphere which solved for z is $z = \sqrt{4 - x^2 - y^2}$. As our surface has the form $z = f(x, y)$ we see that the surface area is $\iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA$. The partial derivatives are $f_x = -x/(\sqrt{4 - x^2 - y^2})$ and $f_y = -y/(\sqrt{4 - x^2 - y^2})$ so $1 + (f_x)^2 + (f_y)^2 = 1 + \frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} = \frac{4}{4 - x^2 - y^2}$. Then $\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{\frac{4}{4 - x^2 - y^2}} = \frac{2}{\sqrt{4 - x^2 - y^2}}$. So the surface area is $\iint_D \frac{2}{\sqrt{4 - x^2 - y^2}}\, dA$. Changing to polar we get

    $$\int_0^{\pi/2} \int_0^1 \frac{2r}{\sqrt{4 - r^2}}\, dr d\theta = \frac{\pi}{2} \int_0^1 \frac{2r}{\sqrt{4 - r^2}}\, dr\ .$$

    u-substitution with $u = 4 - r^2$ gives

    $$-\frac{\pi}{2} \int_4^3 u^{-1/2}\, du = \pi u^{1/2}\Big|_3^4 = \pi(2 - \sqrt{3})\ .$$

15. Evaluate the line integral $\int_C xy\ ds$ where C is the part of the ellipse $x^2 + 4y^2 = 4$ in the first quadrant.

    The ellipse can be parametrized as $x = 2\cos(t), y = \sin(t)$ and the part in the first quadrant corresponds to $0 \le t \le \pi/2$. Then $dx/dt = -2\sin(t), dy/dt = \cos(t)$ so $ds = \sqrt{4\sin^2(t) + \cos^2(t)}dt = \sqrt{3\sin^2(t) + 1}dt$. The line integral is $\int_0^{\pi/2} 2\cos(t)\sin(t)\sqrt{3\sin^2(t)+1}\ dt$. If $u = 3\sin^2(t) + 1$ then $du = 6\sin(t)\cos(t)dt$ so this is $\int_1^4 \frac{1}{3}u^{1/2}\ du = \frac{2}{9}u^{3/2}\Big|_1^4 = 16/9 - 2/9 = 14/9$.

16. Evaluate the surface integral $\iint_S xy\ dS$ where S is the triangular region with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

    The surface is a plane so we will first find the equation of the plane. It contains vectors $\langle -1, 2, 0\rangle$ and $\langle -1, 0, 2\rangle$ so the cross product $\langle 4, 2, 2\rangle$ normal vector to the plane, as is any multiple of this vector including $\langle 2, 1, 1\rangle$. Using the point $(1, 0, 0)$ and normal vector $\langle 2, 1, 1\rangle$ we get that the plane is $2(x - 1) + y + z = 0$ or $2x + y + z = 2$.

    To evaluate the surface integral, we will parameterize the plane as $r(x, y) = \langle x, y, 2 - 2x - y\rangle$. We next need to figure out which values of (x, y) correspond to the triangular region. If we draw the triangular region in 3 dimensions we can see that it's projection down to the xy-plane is the triangle with vertices $(1, 0), (0, 2), (0, 0)$ so the surface integral will be taken over this triangle. If $f(x, y) = 2 - 2x - y$ then $dS = \sqrt{1 + (f_x)^2 + (f_y)^2}dA = \sqrt{1 + (-2)^2 + (-1)^2}dA = \sqrt{6}dA$ so the surface integral $\iint_S xy\ dS = \int_0^1 \int_0^{2-2x} xy\sqrt{6}\ dydx = \sqrt{6}\int_0^1 \frac{1}{2}xy^2\Big|_0^{2-2x}\ dx =$ $\sqrt{6}\int_0^1 \frac{1}{2}x(2 - 2x)^2\ dx = \sqrt{6}\int_0^1 2x^3 - 4x^2 + 2x\ dx = \sqrt{6}(\frac{1}{2}x^4 - \frac{4}{3}x^3 + x^2\Big|_0^1) = \frac{\sqrt{6}}{6}$.

17. Let S be the surface with vector equation $r(u, v) = \langle u\cos(v), u\sin(v), v\rangle$, $0 \le u \le 2$, $0 \le v \le \pi$. See Figure IV on p. 1115 of the textbook for a picture of this surface.

    (a) Find the equation of the plane tangent to S at the point $(0, 1, \pi/2)$.

    First find the values of u, v which correspond to the point $(0, 1, \pi/2)$. These are $u = 1, v = \pi/2$. Then $r_u = \langle\cos(v), \sin(v), 0\rangle$ and $r_v = \langle -u\sin(v), u\cos(v), 1\rangle$ so the cross product is $r_u \times r_v = \langle\sin(v), -\cos(v), u\rangle$. When $u = 1, v = \pi/2$ this is $\langle 1, 0, 1\rangle$ so we are looking for the formula of a plane with point $(0, 1, \pi/2)$ and normal vector $\langle 1, 0, 1\rangle$. The equation of the plane is $x + (z - \pi/2) = 0$ or $x + z = \pi/2$.

    (b) Set up, but do not evaluate, and integral for the surface area of S.

    The surface area is $\int_S 1\ dS$. Then $dS = |r_u \times r_v|dA = \sqrt{\sin^2(v) + \cos^2(v) + u^2}dA = \sqrt{1 + u^2}dA$ and we're integrating over the region on the uv-plane given by $0 \leq u \leq 2$, $0 \leq v \leq \pi$ so we get that the surface area is equal to $\int_0^{\pi} \int_0^2 \sqrt{1+u^2}\ dudv$.

(c) Evaluate the surface integral $\iint_S \sqrt{1+x^2+y^2}\ dS$.

This is the same integral as in part b except we are now integrating $\sqrt{1+x^2+y^2}$ instead of 1. We can the parametric formulas of $r(u,v)$ ($x = u\cos(v), y = u\sin(v),\ z = v$) to convert the thing we're integrating into u's and v's. That is, $\sqrt{1+x^2+y^2} = \sqrt{1+u^2\cos^2(v)+u^2\sin^2(v)} = \sqrt{1+u^2}$. So the surface integral becomes $\int_0^{\pi} \int_0^2 \sqrt{1+u^2}\sqrt{1+u^2}\ dudv = \int_0^{\pi} \int_0^2 1+u^2\ dudv = \frac{14\pi}{3}$.

(d) Evaluate the surface integral $\iint_S F \cdot\ d\mathbf{S}$ where $F = \langle y, x, z^2 \rangle$ and S has upward orientation.

We found the normal vector for S to be $r_u \times r_v = \langle \sin(v), -\cos(v), u \rangle$ which has positive $\mathbf{k}$-component as u is positive so this corresponds to the upward orientation. Then $\iint_S F \cdot\ d\mathbf{S} = \int_0^{\pi} \int_0^2 F(r(u,v)) \cdot (r_u \times r_v)\ dudv = \int_0^{\pi} \int_0^2 \langle u\sin(v), u\cos(v), v^2 \rangle \cdot \langle \sin(v), -\cos(v), u \rangle\ dudv = \int_0^{\pi} \int_0^2 u\sin^2(v) - u\cos^2(v) + v^2 u\ dudv$. We use the trig identity that $\sin^2(v) - \cos^2(v) = -\cos(2v)$ to rewrite this as $\int_0^{\pi} \int_0^2 -u\cos(2v) + v^2 u\ dudv = \int_0^{\pi} -\frac{1}{2}u^2(\cos(2v) - v^2)\Big|_0^2\ dv = \int_0^{\pi} 2v^2 - 2\cos(2v)\ dv = \frac{2}{3}v^3 - \sin(2v)\Big|_0^{\pi} = \frac{2}{3}\pi^3$.