1 Background and context

My research lies primarily in the field of geometric group theory. The origins of this field trace back to Felix Klein, Henri Poincaré, Max Dehn, and many others, who first began a systematic study of groups. The modern "geometric" viewpoint is motivated by ideas of Mikhael Gromov and William Thurston, to name a few. Broadly, geometric group theorists seek to understand (infinite, finitely generated) groups via their presentations by finding well-behaved spaces which encode their symmetry. One then uses the geometry and topology of those spaces to derive algebraic properties of those groups. For me, these well-behaved spaces are assembled using cubes, and the process of associating such a space to a group is called "cubulation," as described in what follows.

1.1 Motivation from the theory of 3-manifolds

The specific source of motivation for the types of questions I wrestle with comes from 3-manifold topology. For many decades, topologists had been interested in completely understanding closed 3-dimensional manifolds. A classification of all such manifolds had seemed out of reach, but William Thurston boldly outlined a program to understand them ([14]). In fact, the 3-manifolds which are hyperbolic (admit a Riemannian metric of everywhere-negative sectional curvature) proved to be particularly difficult to classify. It took 30 years, but in 2012, Ian Agol finally proved the longstanding "Virtual Haken Conjecture" (VHT), thereby placing the last piece of the puzzle to understand these 3-manifolds and realize Thurston's vision ([1]).

Many of the core mathematical ideas that went into Agol's proof depended upon a theory of Daniel Wise, who had been studying objects called CAT(0) cube complexes for some time. A CAT(0) cube complex is a simply-connected and nonpositively curved topological space which is built by gluing cubes of various dimensions together along their faces.

A hyperbolic 3-manifold group G, that is, the fundamental group of a hyperbolic 3-manifold, has a properly discontinuous and cocompact action on hyperbolic 3-dimensional space, which we denote by $\mathbb{H}^3$. Though this action has some desirable properties, it doesn't give enough information about G to prove the VHT on its own. Agol and Wise used a construction due to Michah Sageev from 1995 ([12]) to turn the action of G on $\mathbb{H}^3$ into a proper, cocompact action of G on a "dual" CAT(0) cube complex C built by partitioning $\mathbb{H}^3$ using appropriate "walls" coming from quasi-convex surface subgroups. An interesting feature of C is that it may contain cubes of arbitrarily large dimension. In some ways, this makes C more complex than $\mathbb{H}^3$, but what make C easier to work with is that the walls in C coming from unions of midcubes are easier to work with than the walls coming from the surface subgroups in $\mathbb{H}^3$. It

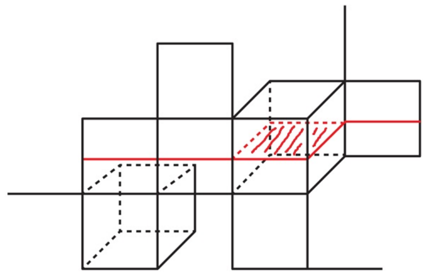

Figure 1: An example of a very small CAT(0) cube complex made of 0, 1, 2, and 3-dimensional cubes. The red subcomplex represents a natural wall, made of "midcubes," inside this space. Image source.

was by working with these walls of C in a hands-on way which allowed Agol to finally prove the Virtual Haken Conjecture.

The resolution of VHT brought CAT(0) cube complexes into the awareness of mathematicians all over the world. It turns out that simply knowing that a group admits a proper, cocompact action on such a space is sufficient to unlock a good deal of structural information about that group. For this reason, "cubulating" groups has become an important goal of modern geometric group theory. Here are just a few of the properties enjoyed by a cubulable group G:

- G satisfies a quadratic isoperimetric inequality, which implies a polynomial-time solution to the word problem in G. In fact, this holds for groups which act on spaces which are CAT(0) in the sense of Alexandrov geometry ([2]).
- G satisfies the Haagerup property, also known as a-T-amenability. Such groups satisfy the Baum-Connes and Novikov conjectures, and also admit uniform embeddings into a Hilbert space ([5], [3]).
- G satisfies a Tits alternative ([13]).
- If G is the fundamental group of a Kähler manifold, then G is virtually a direct product of surface groups, possibly with a free abelian factor ([4]).

1.2 What other groups are cubulable?

Many other groups besides hyperbolic 3-manifold groups have been successfully cubulated. In 2004, Daniel Wise showed that $C'(\frac{1}{6})$ small cancellation groups are cubulable ([15]). An extension of Wise's result for $C'(\frac{1}{6})$ groups was pursued by Alexandre Martin and Markus Steenbock in 2014 when they successfully cubulated $C'(\frac{1}{6})$ small cancellation free products of cubulable groups ([10]). In 2017, Kasia Jankiewicz and Daniel Wise gave a simpler proof of Martin and Steenbock's result relying on Wise's cubical small cancellation theory developed in [16], though they only proved it for $C'(\frac{1}{20})$ small cancellation free products ([8]).

Another candidate class of well-behaved groups are the one-relator groups, groups which admit a presentation of the form $\langle a_1, \ldots, a_m \mid w^n \rangle$ where w is cyclically reduced and not a proper power. If $n \geqslant 6$ these groups are $C'(\frac{1}{6})$ anyway, and if $n = 1$ they need not be cubulable (e.g.

many Baumslag-Solitar groups), so the interesting exponent range is $2 \leqslant n \leqslant 5$. In 2013, Joseph Lauer and Daniel Wise demonstrated one way to cubulate these groups by building a system of nicely-behaved walls in the universal cover of a presentation 2-complex for the group and invoking the Sageev construction ([9]). Their methods work when $n \geqslant 4$. A natural question is whether their result can be generalized to the free product setting.

2 One-relator quotients of free products

Specifically, one can ask whether a one-relator quotient of a free product of cubulable groups ("one-relator product" for short) $G = A * B/\langle\langle w^n \rangle\rangle$ is itself cubulable, where w is a cyclically reduced word of length at least 2 in $A * B$ which is not a proper power, and $n \geqslant 2$.

To tackle this question, we first build a model space X for G by starting with a dumbell space $X_A \vee X_B$ of NPC cube complexes with $\pi_1(X_A) = A$ and $\pi_1(X_B) = B$, and then attaching a 2-cell corresponding to w^n, so that $\pi_1(X) = G$. The task, then, is to build nicely-behaved walls in the universal cover.

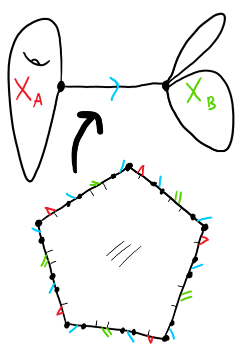

Figure 2: A presentation complex for G. The boundary path of the pentagonal cell corresponds to a word of the form w^5.

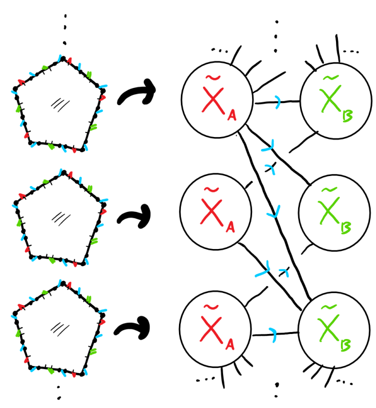

Figure 3: The universal cover of this presentation complex. We build our walls in this space by combining the Lauer-Wise walls considered in [9] with natural cube complex hyperplanes in the factors.

A challenge to overcome is that G is not a Gromov hyperbolic group, in general. The fact that $C'(\frac{1}{6})$ and one-relator groups with torsion are hyperbolic was used critically in [15] and [9] to get that the action of G on the dual cube complex is cocompact, in part because quasiconvexity is much easier to characterize in hyperbolic groups. This was also a concern for Martin and Steenbock. However, one-relator products of locally indicable groups with torsion, like small cancellation free products, are hyperbolic *relative* to the factors, and there is a theorem of Hruska and Wise which helps to achieve cocompactness even in this more general setting ([6], Theorem 7.12).

Though we suspect that the action of G on its dual cube complex is proper and cocompact when $n \geqslant 2$, we unfortunately find it necessary to impose the restriction that $n \geqslant 4$, just as Lauer and Wise do, when seeking to prove that the action is proper. In contrast to Lauer and Wise's setting, it also appears that the condition that $n \geqslant 4$ is necessary for the cocompactness argument. Further work is required to prove that G is cubulable when $n \in \{2, 3\}$.

A group is *locally indicable* if every finitely generated subgroup admits $\mathbb{Z}$ as a homorphic image. This is the loosest well-understood assumption one can put in place to ensure that the natural homomorphisms $A \to G$ and $B \to G$ are injective. We have the following theorem.

Theorem 2.1. *Let $G = A * B/\langle\langle w^n \rangle\rangle$, where A and B are locally indicable. Suppose that w is cyclically reduced, not a proper power, and length at least 2, and that $n \geqslant 4$. Suppose that A and B act properly and cocompactly on CAT(0) cube complexes. Then G admits a proper, cocompact action on a CAT(0) cube complex.*

3 Further directions

In the hyperbolic 3-manifold setting, a significant portion of the story to prove VHT involved not only showing that a hyperbolic 3-manifold group G is cubulable, but that the cube complex C on which it acts has the property of being "virtually special." This condition was originally defined by Wise in terms of certain hyperplane pathologies avoided in some finite cover of the quotient of C by the G-action, and it is not immediately clear that it is a property of *groups*. However, we state the following equivalent definition:

Definition 3.1. *A group G is* virtually special *if it has a finite index subgroup which embeds in a RAAG.*

Ian Agol's fundamental contribution was that any cubulable group which is hyperbolic is virtually special. Beyond the properties enjoyed by cubulable groups listed in section 1.1, a virtually special group G enjoys the following properties (see, e.g., [17]):

- G is residually finite.
- If G is hyperbolic, the quasi-convex subgroups of G are separable.
- G is linear (also see [7]).

Thus the following is a natural question:

Question 3.2. *Let w be a cyclically reduced word of length at least 2 which is not a proper power in $A * B$, and $n \geqslant 2$. Under what circumstances is the group $G = A * B/\langle\langle w^n \rangle\rangle$ virtually special?*

An obvious necessary condition is that the factors A and B must be virtually special. We also suspect it is necessary to assume that the maps $A \to G$ and $B \to G$ are injective, so local indicability of A and B will probably be needed.

Local indicability also implies that G is hyperbolic relative to $\{A, B\}$. Thus if A and B are hyperbolic themselves, then so is G (Corollary 2.41 of [11]), and Agol's theorem implies the following:

Theorem 3.3. *Suppose that A and B are locally indicable, hyperbolic, and cubulable. Let w be a cyclically reduced word of length at least 2 which is not a proper power in $A * B$, and $n \geqslant 2$. Then $G = A * B/\langle\langle w^n \rangle\rangle$ is virtually special.*

The following conjecture is natural:

Conjecture 3.4. *Suppose that A and B are locally indicable and virtually special. Let w be a cyclically reduced word of length at least 2 which is not a proper power in $A * B$, and $n \geqslant 2$. Then $G = A * B/\langle\langle w^n \rangle\rangle$ is virtually special.*

One strategy of proof would be to generalize Wise's theory of *quasiconvex heirarchies* to the relatively hyperbolic setting. This approach can be used to directly show the following strong generalization of cubulability of one-relator groups with torsion, as outlined by Wise in [16]:

Theorem 3.5. *Let $H = \langle a_1, \ldots, a_m \mid w^n \rangle$ where w is cyclically reduced and not a proper power, and $n \geqslant 2$. Then H is virtually special.*

We end with a final question in a slightly different direction. One could also consider small cancellation quotients of free products of virtually special groups. As remarked in section 1.2, Martin and Steenbock show that a $C'(\frac{1}{6})$ quotient of $A * B$, where A and B are cubulable, is itself cubulable. The natural question is the following:

Question 3.6. *Let G be a $C'(\frac{1}{6})$ quotient of $A * B$, where A and B are virtually special. Is G virtually special?*

Perhaps it is conceivable that a suitable theory of quasi-convex heirarchies in the relatively hyperbolic setting could be used to tackle this question as well.

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