1 Background and context

My research lies primarily in the field of geometric group theory. The origins of this field trace back to Felix Klein, Henri Poincaré, Max Dehn, and many others, who first began a systematic study of groups. The modern "geometric" viewpoint is motivated by ideas of Mikhael Gromov and William Thurston, to name a few. Broadly, geometric group theorists seek to understand (infinite, finitely generated) groups via their presentations by finding well-behaved spaces which encode their symmetry. One then uses the geometry and topology of those spaces to derive algebraic properties of those groups. For me, these well-behaved spaces are assembled using cubes, and the process of associating such a space to a group is called "cubulation," as described in what follows.

1.1 Motivation from the theory of 3-manifolds

The specific source of motivation for the types of questions I wrestle with comes from 3-manifold topology. For many decades, topologists had been interested in finding a scheme to understand the possible geometries of closed 3-dimensional manifolds. A classification of all such geometries had seemed out of reach, but William Thurston boldly outlined a program to understand them [10]. The 3-manifolds which are hyperbolic (admit a Riemannian metric of constant negative sectional curvature) proved to be particularly difficult to classify. It took 30 years, but in 2012, Ian Agol and Daniel Wise, building on the work of many others, finally proved the longstanding "Virtual Haken Conjecture" (VHC), thereby placing the last piece of the puzzle of a realization of a large portion of Thurston's vision [1].

Much of the theory that went into the proof of the VHC was developed by Wise, who had been studying objects called CAT(0) cube complexes for some time. A CAT(0) cube complex is a simply-connected and nonpositively curved topological space which is built by gluing cubes of various dimensions together along their faces. The proof of the VHC involves replacing a given hyperbolic 3-manifold with a CAT(0) cube complex on which the fundamental group of that manifold acts, using a construction of Michah Sageev [8]. It was by working with this cube complex (and its natural "walls;" see Figure 1) in a hands-on way which allowed Agol to finish the proof of the VHC.

The resolution of the VHC brought cube complexes into the awareness of mathematicians all over the world. As it turns out, simply knowing that a group admits a proper, cocompact action on a CAT(0) cube complex is sufficient to unlock a good deal of structural information about that group. Thus, "cubulating" groups has become an important goal of modern geometric group theory. Here are just a few of the properties enjoyed by a cubulable group G:

- G satisfies a quadratic isoperimetric inequality, which implies a polynomial-time solution

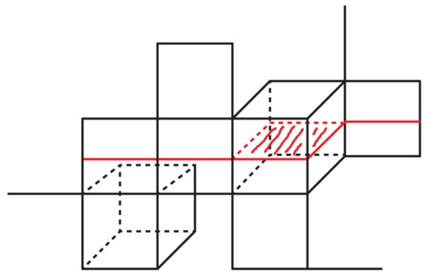

Figure 1: An example of a very small CAT(0) cube complex made of 0, 1, 2, and 3-dimensional cubes. The red subcomplex represents a natural wall, made of "midcubes," inside this space. Image source.

to the word problem in G. In fact, this holds for groups which act on spaces which are CAT(0) in the weaker sense of Alexandrov geometry [2].

- G satisfies a Tits alternative – it either contains a free subgroup of rank 2 or is virtually abelian [9].
- If G is the fundamental group of a Kähler manifold, then G is virtually a direct product of surface groups, possibly with a free abelian factor [3].

1.2 What other groups are cubulable?

Many other groups besides hyperbolic 3-manifold groups have been successfully cubulated. In 2004, Daniel Wise showed that $C'(\frac{1}{6})$ small cancellation groups are cubulable [11]. An extension of Wise's result for $C'(\frac{1}{6})$ groups was pursued by Alexandre Martin and Markus Steenbock in 2014 when they successfully cubulated $C'(\frac{1}{6})$ small cancellation free products of cubulable groups [7].

Another candidate class of well-behaved groups are the one-relator groups, groups which admit a presentation of the form $\langle a_1, \ldots, a_m \mid w^n \rangle$ where w is cyclically reduced and not a proper power. If $n \geqslant 6$ these groups are $C'(\frac{1}{6})$ anyway, and if $n = 1$ they need not be cubulable (e.g. many Baumslag-Solitar groups), so the interesting exponent range is $2 \leqslant n \leqslant 5$. In 2013, Joseph Lauer and Daniel Wise demonstrated one way to cubulate these groups by building a system of nicely-behaved walls in the universal cover of a presentation 2-complex for the group and invoking the Sageev construction [5]. Their methods work when $n \geqslant 4$. A natural question is whether their result can be generalized to the free product setting.

2 One-relator quotients of free products

Specifically, one can ask whether a one-relator quotient of a free product of cubulable groups ("one-relator product" for short) $G = A * B / \langle\langle w^n \rangle\rangle$ is itself cubulable, where w is a cyclically reduced word of length at least 2 in $A * B$ which is not a proper power, and $n \geqslant 2$.

To tackle this question, we first build a model space X for G by starting with a dumbell space $X_A \vee X_B$ of nonpositively curved cube complexes with $\pi_1(X_A) = A$ and $\pi_1(X_B) = B$, and then attaching a 2-cell corresponding to w^n, so that $\pi_1(X) = G$. The task, then, is to build nicely-behaved walls in the universal cover.

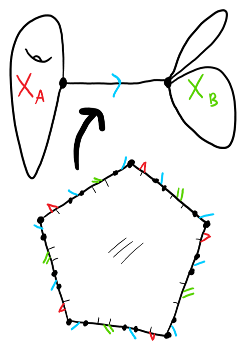

Figure 2: A presentation complex for G. The boundary path of the pentagonal cell corresponds to a word of the form w^5.

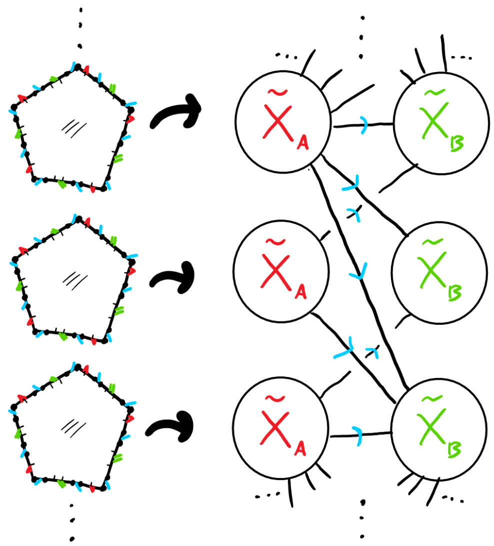

Figure 3: The universal cover of this presentation complex. We build our walls in this space by combining the Lauer-Wise walls considered in [5] (in the pentagonal cells) with the natural hyperplanes in the CAT(0) cube complex factors $\tilde{X}_A$ and $\tilde{X}_B$.

A group is *locally indicable* if every finitely generated subgroup admits $\mathbb{Z}$ as a homomorphic image. This is the loosest well-understood assumption one can put in place to ensure that the natural homomorphisms $A \to G$ and $B \to G$ are injective. We have the following theorem.

Theorem 2.1. *Let A and B be locally indicable groups, w a word in $A * B$ which is not conjugate into A or B, and $n \geqslant 4$. Then $G = A * B / \langle\langle w^n \rangle\rangle$ is cubulable if A and B are.*

3 Further directions

In the hyperbolic 3-manifold setting, a significant portion of the story to prove the VHC involved not only showing that a hyperbolic 3-manifold group G is cubulable, but that the cube complex on which it acts has the property of being "virtually special," i.e., it has a finite index subgroup which embeds in a right-angled Artin group (RAAG).

Beyond the properties enjoyed by cubulable groups listed in section 1.1, a virtually special group G enjoys the following properties (see, e.g., [13]):

- G is residually finite.
- If G is hyperbolic, the quasiconvex subgroups of G are separable.
- G is linear (also see [4]).

The following would be a natural follow-up to Theorem 2.1:

Conjecture 3.1. *Suppose that A and B are locally indicable and virtually special. Let w be a word in $A * B$ which is not conjugate into A or B, and $n \geqslant 2$. Then $G = A * B/\langle\langle w^n \rangle\rangle$ is virtually special.*

One strategy of proof would be to generalize Wise's theory of *quasiconvex hierarchies* to the relatively hyperbolic setting. In fact, this has already been done in case the peripheral subgroups are virtually abelian [12]. These hierarchies can be used to directly show the following strong generalization of cubulability of one-relator groups with torsion:

Theorem 3.2. *[12, Corollary 18.2] Let $G = \langle a_1, \ldots, a_m \mid w^n \rangle$ where w is cyclically reduced, and $n \geqslant 2$. Then G is virtually special.*

One could also consider small cancellation quotients of free products of virtually special groups. As remarked in section 1.2, Martin and Steenbock show that a $C'(\frac{1}{6})$ quotient of $A * B$, where A and B are cubulable, is itself cubulable. The natural question is the following:

Question 3.3. *Let G be a $C'(\frac{1}{6})$ quotient of $A * B$, where A and B are virtually special. Is G virtually special?*

Perhaps it is conceivable that a suitable theory of quasiconvex hierarchies in the relatively hyperbolic setting could be used to tackle this question as well.

4 Opportunities for undergraduate involvement

I have experience mentoring undergraduates with a view towards research projects. At the University of Oklahoma, I mentored with a directed reading program which consisted of weekly meetings with undergraduates to discuss graduate-level mathematical topics including algebraic topology and geometric group theory. I assigned weekly readings and exercises to my student, and we met in my office to discuss his questions, many of which I struggled to answer. I became better and better at distilling complex ideas to my student over time, and I think I helped solidify his decision to attend graduate school for mathematics. In the process, I developed a passion for communicating clearly about what mathematics research really is, and since then I have striven to craft explanations of mathematical research not only for mathematics majors, but for any student who might wonder what the day-to-day life of a mathematician is like.

While Conjecture 3.1 and Question 3.3 may not be tractable as undergraduate research projects, there are plenty of interesting questions about one-relator groups and right-angled Artin groups that are quite approachable. For instance, a basic fact about one relator groups is the following "Freiheitssatz," due to Wilhelm Magnus. My proof of Theorem 2.1 depends heavily on the analogue of this statement for one-relator products.

Theorem 4.1. *(see [6]) Let $G = \langle a_1, \ldots, a_m \mid w \rangle$, where w is cyclically reduced and involves at least two of the a_i's. Then any n-element proper subset of the a_i's generates a free subgroup of G of rank n.*

I would very much like to know when the analogous statement holds for RAAGS, i.e.

Question 4.2. *Let G be a RAAG with a presentation of the form $\langle a_1, \ldots, a_m \mid * \rangle$, where $*$ represents some collection of commutators of the a_i's. Let w be a cyclically reduced word in G which involves at least two of the a_i's. Let H be a subgroup of G generated by some proper subset of the a_i's. Under what conditions (on G, H, and w) is the natural homomorphism $f_H : H \to G/\langle\langle w \rangle\rangle$ injective?*

If the f_H maps are always injective, there may be hope to cubulate $G/\langle\langle w \rangle\rangle$ or prove other desirable properties about it. However, I have used a topological approach to produce many examples where f_H is not injective. As a result of working through these examples, I have some feeling for when f_H should be injective. What I would really like to do (and enterprising undergraduates could help!) is to use the GAP computer algebra system to create many examples, test whether f_H is injective, and try to observe a pattern. This could lead to a full or partial answer to Question 4.2. Developing the algebra and topology to understand this problem and its relatives would be an excellent way to expand an undergraduate's knowledge and intuition, prepare them for graduate school, or otherwise cultivate their problem-solving abilities.

Computational approaches

In addition to identifying patterns and coming up with conjectures, I am a believer in using a computational approach to directly solve problems by computer search, and I feel well-equipped to help undergraduates to do the same. As an undergraduate myself, my senior thesis involved exploring a computational approach to raising the lower bound of the Ramsey number $R(5, 5)$, which can be defined as the smallest number N such that for any group of N people, there exist five people in that group who are either mutual strangers or mutual acquaintances. While I was ultimately unsuccessful, the advice and guidance from my undergraduate academic advisers was encouraging, and I learned a great deal in the process. In fact, this project was my first exposure to research in mathematics, and ultimately influenced my decision to attend graduate school.

More recently, I wrote a program to explore how one might raise the lower bound of the chromatic number of the plane, defined to be the smallest number of colors needed to color the Euclidean plane so that no two points distance one apart are the same color. The best lower and upper bounds of 4 and 7 stood for more than 50 years, until amateur mathematician and anti-aging expert Aubrey de Grey (who has a background in computer science) established in April of 2018 that the chromatic number of the plane is at least 5 using a clever algorithm.

In addition to leading reading courses, supervising expository projects, and finding extracurricular research opportunities (such as REU's) for undergraduates, I thus feel prepared to help them to use computational approaches both to gather evidence in support of various conjectures and to prove results by computer search.

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