Frobenius Reciprocity for Jacquet Modules

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We introduce the notion of a \textit{Jacquet Module} and give a proof of Frobenius Reciprocity in this case.

Let $G$ be a locally profinite group and $(\pi, V)$ be a smooth representation of $G$. Fix a parabolic subgroup $P$ of $G$. Let $P = MN$ be a Levi decomposition of $P$ where $M$ is a Levi component and $N$ is the unipotent radical. We know that $N$ is a normal subgroup of $P$ and $P/N \simeq M$. Let $V(N) = span_{\mathbb{C}}\{\pi(n)v - v | n \in N, v \in V\}$.

\textbf{Lemma 1.} $V(N)$ is an invariant subspace for $P$

\textit{Proof.} Let $p \in P$. We have to show that $\pi(p)(\pi(n)v - v) \in P$.

\[\pi(p)(\pi(n)v - v) = \pi(pn)v - \pi(p)v = \pi(pnp^{-1}p)v - \pi(p)v\]

[Since $N \trianglelefteq P$, $pnp^{-1} \in N$]

\[= \pi(n')w - w \quad [n' = pnp^{-1}, w = \pi(p)v]\]

\[\square\]

Let $V_N = V/V(N)$. Since $V(N)$ is invariant under $P$ we get a representation $(\pi_N, V_N)$ of $P$. $(\pi_N, V_N)$ is called the Jacquet Module of $N$ with respect to the parabolic subgroup $P$.

\textbf{Notation:} $(\pi, V)$ is an irreducible representation of $G$ and $(\sigma, W)$ an irreducible representation of $M$. Let $(i_P^G \sigma, \mathcal{M})$ be the parabolically induced representation of $G$ and $(\pi_N, V_N)$ the Jacquet Module of $N$ with respect to $P$. We will look at $\pi_N$ as a representation of $M$ by restriction.

\textbf{Theorem 1. Frobenius Reciprocity for Jacquet Modules}

\[Hom_G(\pi, i_P^G \sigma) \simeq Hom_M(\pi_N, \sigma)\]

\textit{Proof.} Let $\phi$ be the isomorphism between. We will explicitly describe this map $\phi$ and its inverse.

\textbf{Construction of the map $\phi$}

Let $T \in Hom_G(\pi, i_P^G \sigma)$. Define $\hat{T} : V \to W$ as $\hat{T}(v) = T(v)(1)$. We will show that $V(N) \subset ker(\hat{T})$. Therefore $\hat{T}$ factors to a map $\delta_T : V_N \to W$.
**Proof:** $V(N) \subset \ker(\delta_T)$

Let $\pi(n)v - v \in V(N)$. Consider $\delta_T(\pi(n)v - v)$

\[
\begin{align*}
\delta_T(\pi(n)v - v) &= T(\pi(n)v - v)(1) \\
&= T(\pi(n)v)(1) - T(v)(1) \\
&= ((i_G^\pi\sigma)(n)(T(v)))(1) - T(v)(1) \quad [\because T \in \text{Hom}_G(\pi, i_G^G\sigma)] \\
&= T(v)(n) - T(v)(1) \quad [\text{Action of the induced representation is on the right}] \\
&= \sigma(n)T(v)(1) - T(v)(1) \quad [\because T(v) \in \mathfrak{m}] \\
&= T(v)(1) - T(v)(1) \quad [\because \sigma(n) = 1] \\
&= 0
\end{align*}
\]

We will now show that $\delta_T \in \text{Hom}_M(\pi_N, \sigma)$

**Proof:** $\delta_T \in \text{Hom}_M(\pi_N, \sigma)$

We have to show that $\delta_T(\pi_N(m)(v + V(N))) = \sigma(m)(\delta_T(v + V(N)))$

\[
\begin{align*}
\delta_T(\pi_N(m)(v + V(N))) &= \delta_T(\pi(m)v + V(N)) \\
&= \delta_T(\pi(m)v) \\
&= T(\pi(m)v)(1) \quad [\text{By definition of the map}] \\
&= (i_G^G)(m)(T(v))(1) \quad [\because T \in \text{Hom}_G(\pi, i_G^G\sigma)] \\
&= T(v)(m) \quad [\text{Action of the induced representation is on the right}] \\
&= \sigma(m)T(v)(1) \quad [\because T(v) \in \mathfrak{m}] \\
&= \sigma(m)(\delta_T(v)) \\
&= \sigma(m)(\delta_T(v + V(N)))
\end{align*}
\]

**Definition of the map $\phi$**

We define $\phi : \text{Hom}_G(\pi, i_G^G\sigma) \to \text{Hom}_M(\pi_N, \sigma)$ as $\phi(T) = \delta_T$

**Construction of the map $\phi^{-1}$**

Let $S \in \text{Hom}_M(\pi_N, \sigma)$. Define $I_S : V \to W$ as $I_S(v)(g) = \dot{S}(\pi(g)v)$ where $\dot{S} : V \to W$ is defined as $\dot{S}(v) = S(v + V(N))$. We will show that $I_S \in \text{Hom}_G(\pi, i_G^G\sigma)$.

**Proof:** $I_S \in \text{Hom}_G(\pi, i_G^G\sigma)$

\[
\begin{align*}
(i_G^G\sigma)(g)(I_S(v))(h) &= I_S(v)(hg) \\
&= \dot{S}(\pi(hg)v) \\
&= I_S(\pi(g)v)(h)
\end{align*}
\]
**Definition of the map $\phi^{-1}$**

We define $\phi^{-1}: \text{Hom}_M(\pi_N, \sigma) \rightarrow \text{Hom}_G(\pi, i^G_P \sigma)$ as $\phi^{-1}(S) = I_S$

**Conclusion**

We see that $\phi$ is clearly an isomorphism between $\text{Hom}_G(\pi, i^G_P \sigma)$ and $\text{Hom}_M(\pi_N, \sigma)$. i.e $\phi$ is a linear map and satisfies $\phi^{-1}(\phi(T)) = T$ and $\phi(\phi^{-1}(S)) = S$. In other words $I_{\delta_T} = T$ and $\delta_{I_S} = S$.