# Research Statement

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## 1. Introduction

Let $G$ be a topological group and $\pi$ be an irreducible representation of $G$ such that $\pi \simeq \pi^\vee$ ($\pi^\vee$ denotes the dual or contragredient of $\pi$.) In this situation, the underlying space $V$ of $\pi$ admits a non-degenerate $G$-invariant bilinear form. This form is unique up to scalars (by Schur's Lemma) and is easily seen to be symmetric or skew-symmetric. Accordingly, we define

$$\varepsilon(\pi) = \begin{cases} 1 & \text{if the form is symmetric} \\ -1 & \text{if the form is skew-symmetric} \end{cases}$$

and call it the sign of $\pi$. Computing $\varepsilon(\pi)$ is an old problem that has been extensively studied for finite groups and compact Lie groups ([1, 4, 7, 9]). In this setting, the sign is sometimes referred to as the Frobenius-Schur indicator of $\pi$. For reductive $p$-adic groups, there is a small but growing literature on signs with a lot of interesting problems to explore (see, e.g., [6, 10, 11] ). For my doctoral research, I have focused on studying the sign for a certain class of representations of a reductive $p$-adic group $G$. To be more precise, I have studied the sign when $\pi$ is a generic representation of $G$ with non-zero vectors fixed under an Iwahori subgroup.

The motivation for this problem came from the work of Prasad [10]. He showed that, for $\pi$ an irreducible smooth generic representation of a quasi-split reductive $p$-adic group $G$, the sign $\varepsilon(\pi)$ is determined by evaluating the central character $\omega_\pi$ of $\pi$ at a special element in the center of the group, whenever it exists. He uses Galois cohomology to show the existence of this special central element, when $G$ is a finite group of Lie type with a connected center. In my thesis, I have used techniques independent of his methods to prove the existence of the special central element for a split reductive $p$-adic group $G$ with a connected center and used it to compute the sign of $\pi$ in the Iwahori fixed case.

## 2. Basic Definitions

Let $F$ be a non-Archimedean local field. We write $\mathfrak{O}$ for the ring of integers in $F$, $\mathfrak{p}$ for the unique maximal ideal in $\mathfrak{O}$ and $k = \mathfrak{O}/\mathfrak{p}$ for the residue field. We also fix a Borel subgroup $B$ defined over $F$. We write $U$ for the unipotent radical of $B$ and $U_\alpha$

for the root subgroups corresponding to the simple roots $\alpha$ of $T$ in $B$ ($T$ is a maximal torus in $B$).

**Definition 2.1.** The Iwahori subgroup $I$ is defined to be the inverse image of $B(k)$ ($k$-points of $B$) under the canonical map (reduction mod $\mathfrak{p}$) $G(\mathfrak{O}) \to G(k)$.

**Example 2.2.** Let $G = \mathrm{GL}(n, F)$. Take $B$ to be the standard Borel subgroup (upper triangular matrices) and $U$ to be the unipotent radical (unipotent matrices) of $B$ in $G$. In this case the Iwahori subgroup is the collection of matrices of the following type.

$$I = \begin{pmatrix} \mathfrak{O}^\times & \mathfrak{O} & \cdots & \mathfrak{O} \\ \mathfrak{p} & \mathfrak{O}^\times & \cdots & \mathfrak{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \cdots & \mathfrak{O}^\times \end{pmatrix}$$

**Definition 2.3.** A character $\psi$ of $U$ is non-degenerate if $\psi|_{U_\alpha} \neq 1$ for all simple roots $\alpha$.

**Example 2.4.** Let $G = \mathrm{GL}(n, F)$. In this case, any non-degenerate character $\psi$ of $U$ is given by

$$\psi(u) = \theta(\alpha_1 u_{12} + \alpha_2 u_{23} + \cdots + \alpha_{n-1} u_{n-1n})$$

where $\theta$ is a complex valued non-trivial additive character of $F$, $u = (u_{ij})$ and $\alpha_1, \ldots, \alpha_n \in F^\times$.

**Definition 2.5.** A representation $\pi$ is called generic if there exists a non-degenerate character $\psi$ of $U$ such that $\mathrm{Hom}_U(\pi, \psi) \neq 0$.

## 3. Main result from Thesis

I will state the main result from my thesis and briefly summarize the steps used in the proof. Throughout this section, we take $G$ to be the group of $F$-points of a split connected reductive $F$-group over a non-Archimedean local field $F$ of characteristic 0 with a maximal $F$-split torus $T$.

**Theorem 3.1.** *Let $\pi$ be an irreducible smooth self-dual representation of $G$ with non-zero vectors fixed under an Iwahori subgroup. If $\pi$ is also generic, then $\varepsilon(\pi) = 1$.*

We consider the following cases to prove the main result.

1) Center of $G$ is connected.
2) Center of $G$ is not connected.

When the center of $G$ is connected, we show that there exists an element $s \in T(\mathfrak{O})$ ($\mathfrak{O}$-points of $T$) such that $\alpha(s) = -1$ for all simple roots $\alpha$. Using an elegant idea of Rodier [13], we get a compact open subgroup $K_m$ and a character $\psi_m$ of $K_m$ such that $\dim \mathrm{Hom}_{K_m}(\pi, \psi_m) = 1$. It is easy to see that the element $s \in T(\mathfrak{O})$ satisfies the hypotheses of Prasad's Theorem ([10] Proposition 2) (i.e., $s$ normalizes $K_m$ and inner conjugation by $s$ takes the character $\psi_m$ to its inverse). Using the fact that $\pi$ has Iwahori fixed vectors and $T(\mathfrak{O})$ is a subset of the Iwahori subgroup it follows that $\omega_\pi(s^2) = 1$ and hence $\varepsilon(\pi) = 1$.

When the center of $G$ is not connected, we construct a split connected reductive $F$-group $\tilde{G}$ with a maximal $F$-split torus $\tilde{T}$ so that $G$ imbeds as a subgroup of $\tilde{G}$. This group $\tilde{G}$ has a connected center $\tilde{Z}$. We show that there exists a self-dual character $\nu$ of $\tilde{Z}$ extending the central character $\omega_\pi$. Using this character $\nu$, we extend $\pi$ to a representation of $\tilde{Z}G$. We show that the quotient group $\tilde{G}/(\tilde{Z}G)$ is a finite abelian group. We use the results of Gelbart and Knapp [2] to get an irreducible representation $\tilde{\pi}$ of $\tilde{G}$ containing $\pi$. In fact, $\tilde{\pi}$ is self-dual up to a twist by a character $\chi$ of $\tilde{G}$. We prove that $\tilde{\pi}$ can be chosen to have non-zero vectors fixed under an Iwahori subgroup in $\tilde{G}$. Since center of $\tilde{G}$ is connected we get an element $s \in \tilde{T}(\mathfrak{O})$ satisfying the hypotheses of Prasad's Theorem. We prove that $\varepsilon(\tilde{\pi}) = \chi(s)\omega_{\tilde{\pi}}(s^2)$ and $\varepsilon(\tilde{\pi}) = \varepsilon(\pi)$. Finally using the fact that $\tilde{\pi}$ has Iwahori fixed vectors we show that $\chi(s) = \omega_{\tilde{\pi}}(s^2) = 1$.

## 4. Future Research

The main result (Theorem 3.1) was proved under the assumption that the representation $\pi$ is generic. Genericity is a technical condition imposed on $\pi$, in order to use the techniques of Prasad in studying the sign. It should be possible to prove the main result (Theorem 3.1) for $\pi$ a non-generic representation of $G$. My immediate research goal would be to try and establish the result for the non-generic case.

**Conjecture 1.** *Let $\pi$ be an irreducible smooth self-dual representation of $G$ with non-zero vectors fixed under an Iwahori subgroup. Then $\varepsilon(\pi) = 1$.*

The key ideas for proving the main result (Theorem 3.1) came from studying the sign of an irreducible generic representation of $\mathrm{SL}(n, F)$. Currently, I am trying to establish Conjecture 1 for $\mathrm{SL}(n, F)$. I hope that completing the problem for $\mathrm{SL}(n, F)$ will give some insight into proving the result for any arbitrary group $G$.

Another short term research goal would be to extend the main result (Theorem 3.1) for a split or quasi-split reductive group $G$ defined over a field $F$ of arbitrary characteristic. To be more precise, Let $F$ be a non-Archimedean local field of arbitrary characteristic and $G$ be the group of $F$-points of a split or quasi split connected reductive $F$-group. We are interested in proving the following result.

**Conjecture 2.** *Let $\pi$ be an irreducible smooth self-dual representation of $G$ with non-zero vectors fixed under an Iwahori subgroup. If $\pi$ is generic, then $\varepsilon(\pi) = 1$.*

4.1. **Twisted signs and representations.** The notion of twisted signs was introduced by Kawanaka and Matsuyama [5] in 1990. I will briefly recall the notion of twisted signs and outline the problems I hope to study in future.

Let $G$ be a topological group and let $\theta$ be a continuous automorphism of $G$ of order at most two. Suppose that $\pi$ is an irreducible representation of $G$ such that $\pi^\theta \simeq \pi^\vee$ where $\pi^\theta$ the $\theta$-twist of $\pi$ given by $\pi^\theta(g) = \pi(\theta(g))$, for $g \in G$. The underlying space $V$ of $\pi$ admits a non-degenerate bilinear form $(\cdot,\cdot)$ which is $G$-invariant in the following sense.

$$(\pi(g)v_1, \pi^\theta(g)v_2) = (v_1, v_2), \quad g \in G,\, v_1, v_2 \in V.$$

We can show that this form is either symmetric or skew-symmetric and define

$$\varepsilon_\theta(\pi) = \begin{cases} 1 & \text{if the form is symmetric} \\ -1 & \text{if the form is skew-symmetric} \end{cases}$$

If $\theta$ is an involution (i.e., has order two), we refer to $\varepsilon_\theta(\pi)$ as the $\theta$-twisted sign of $\pi$.

**Remark 4.1.** If $\theta = 1$, the identity map on $G$ then $\pi$ is self-dual and we simply write $\varepsilon(\pi)$ in place of $\varepsilon_1(\pi)$. In this sense, the twisted sign is a generalization of the usual sign.

I am interested in studying the twisted sign for the groups of $F$-points of certain reductive algebraic groups over $F$. Most groups $G$ in this class admit an involution $\theta$ such that $\pi^\theta \simeq \pi^\vee$, for all irreducible representations $\pi$ of $G$. For example, if $G = GL_n(F)$ then it is an old result of Gelfand and Kazhdan [3] that the involution $g \xrightarrow{\theta} ({}^\top g)^{-1}$ (where $\top$ denotes *transpose*) has this property. It has been recently proved that the twisted sign $\varepsilon_\theta(\pi)$ ($\theta$ is the transpose inverse involution) is always 1 in the case of $\mathrm{GL}(n, F)$ (see, e.g., [14, 12] ). Further, Waldspurger has observed that other classical groups over $F$ also admit such involutions ([8] Chap. 4.II.1). I am interested in understanding the twisted signs for these other classical groups.

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