University of Oklahoma
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Final Project

Lie Theory

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December 2013

Abstract

This document contains some of the important definition and theorems taught in the course of Lie Theory - I (Fall 2013 at the University of Oklahoma). Mostly class notes and the text book "Lie Groups, Lie Algebras, and Representations: An Elementary Introduction" by Hall,B. [1] were used in writing these notes. No proofs are given except in some cases in the chapter [3] and it is essentially the same material as in [1]. Chapter [2] contains the material from Homework Assignment II and the proof can be found in [2].

Contents

Chapter 1

Matrix Lie Groups

1.1 Introduction

Matrix Groups are important examples of Lie Groups. Therefore it makes sense to study these concrete examples before embarking on the study of more general Lie Groups.

1.2 Matrix Lie Group

Definition 1.1. Let us denote any of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ by $\mathbb{F}$ and the set of all $n \times n$ matrices over $\mathbb{F}$ by $M_n(\mathbb{F})$, i.e.

$$M_n(\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} : a_{i,j} \in \mathbb{F} \right\}$$

.

This is a vector space over $\mathbb{F}$ of dimension n^2. It is easy to see, for example, that

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \dots$$

form a basis of $M_n(\mathbb{F})$.

1.2.1 The General Linear Group

Definition 1.2. The general linear group over a field $\mathbb{F}$, denoted by $GL_n(\mathbb{F})$ is the group of all $n \times n$ matrices with the entries from the field $\mathbb{F}$.

Proposition 1.3. *$GL_n(\mathbb{F})$is an open subset of $M_n(\mathbb{F})$*

Definition 1.4. A Matrix Lie Group is a closed subgroup of $GL_n(\mathbb{F})$ for some n and $\mathbb{F}$.

1.2.2 Examples of Matrix Lie Group :

$GL_n(\mathbb{C})$ - We denote by $GL_n(\mathbb{C})$ the complex general linear group consisting of all nonsingular n-by-n complex matrices with matrix multiplication as group operation.

$GL_n(\mathbb{R})$ - $GL_n(\mathbb{R})$ is the real general linear group consisting of all nonsingular n-by-n real matrices with matrix multiplication as group operation.

$GL_n(\mathbb{H})$ - Let us define

$$\mathbb{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

then $\mathbb{H}$ is group of Quaternions.The general linear group over Quaternions is -

$$GL_n(\mathbb{H}) = \left\{ \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \cdots & \cdots & \cdots \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

$O_n(\mathbb{C})$ - Orthogonal Group over Complex numbers

$$O_n(\mathbb{C}) = \{g \in GL_n(\mathbb{C}) : gg^T = I\}$$

$O_n(\mathbb{R})$ - Orthogonal Group over Reals

$$O_n(\mathbb{R}) = \{g \in GL_n(\mathbb{R}) : gg^T = I\}$$

$Sp_{2n}(\mathbb{C})$ - Sympleptic Group over Complex numbers

Let I_n denote the unit matrix of order n. Let use define

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

then

$$Sp_{2n}(\mathbb{C}) = \{g \in GL_n(\mathbb{C}) : gJg^T = J\}$$

$Sp_{2n}(\mathbb{R})$ - Sympleptic Group over Reals

$$Sp_{2n}(\mathbb{R}) = \{g \in GL_n(\mathbb{R}) : gJg^T = J\}$$

Let us define

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

Now we have:

$$O_{p,q}(\mathbb{R}) = \{g \in GL_{p+q}(\mathbb{R}) : gI_{p,q}g^T = I_{p,q}\}$$

$$U_{p,q}(\mathbb{C}) = \{g \in GL_{p+q}(\mathbb{C}) : gI_{p,q}\bar{g}^T = I_{p,q}\}$$

Chapter 2

The Classical Groups

2.1 Isometry Groups of Bilinear Forms

2.1.1 Definition : Non Degenerate Alternate Form

Definition 2.1. Let F be a field and let V be a finite dimensional vector space over F. An alternate form on V is a function

$$V \times V \ni u, v \to (u, v) \in F$$

such that for all $u, u', v \in V$ and $a \in \mathbb{F}$

$$\begin{aligned}(u, u) &= 0\\ (u + u', v) &= (u, v) + (u', v)\\ (au, v) &= a(u, v) = (u, av)\end{aligned}$$

The form (,) is called non-degenerate if $(u, v) = 0$ for all $v \in V$ implies $u = 0$:

2.1.2 Fact: Matrix of Non Degenerate Alternate Form

Proposition 2.2. *Suppose (,) is non-degenerate alternate form on V. Then there is a basis $v_1, v_2, \ldots, v_{2n}$ of V such that the matrix associated to this form*

$$((v_i, v_j))_{1 \leq i,j \leq 2n} = diag(J, J, \ldots, J) \qquad where\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In particular the dimension of V is even.

2.1.3 Definition : Non Degenerate Symmetric Form

Definition 2.3. Let F be a field and let V be a finite dimensional vector space over F. A Symmetric form on V is a function

$$V \times V \ni u, v \to (u, v) \in F$$

such that for all $u, u', v \in V$ and $a \in \mathbb{F}$

$$(u, v) = (v, u)$$
$$(u + u', v) = (u, v) + (u', v)$$
$$(au, v) = a(u, v)$$

The form (,) is called non-degenerate if $(u, v) = 0$ for all $v \in V$ implies $u = 0$:

2.1.4 Fact: Matrix of Non Degenerate Symmetric Form

Proposition 2.4. *Suppose (,) is non-degenerate symmetric form on V. Assume that the Field F is of characteristic $\neq 2$. Then there is a basis Then there is a basis $v_1, v_2, \ldots, v_{2n}$ of V such that the matrix associated to this form*

$$((v_i, v_j))_{1 \leq i,j \leq 2n} = diag(b_1, b_2, \ldots, b_n) where\ b_1, b_2, \ldots, b_n \in \mathbb{F} - 0$$

In particular, if F is algebraically closed then we may assume that $b_i = 1$ for $i = 1, 2 \ldots, n$ If $\mathbb{F} = \mathbb{R}$, then we may assume that $n = p + q$, $b_1 = b_2 = \ldots = b_p = 1$ and $b_{p+1} = b_{p+2} = \ldots = b_n = -1$. The difference $p - q$ is called the signature of the form

2.1.5 Definition : Non Degenerate Hermitian Form

Definition 2.5. Let F be a field and let V be a finite dimensional vector space over F. A Hermitian form on V is a function

$$V \times V \ni u, v \to (u, v) \in F$$

such that for all $u, u', v \in V$ and $a \in \mathbb{F}$

$$(u, v) = \overline{(v, u)}$$
$$(u + u', v) = (u, v) + (u', v)$$
$$(au, v) = a(u, v)$$

The form (,) is called non-degenerate if $(u, v) = 0$ for all $v \in V$ implies $u = 0$:

2.1.6 Fact: Matrix of Non Degenerate Hermitian Form

Proposition 2.6. *Suppose (,) is non-degenerate hermitian form on V. Then there are nonnegative integers* p, q*, such that* $n = p + q$ *and Then there is a basis Then there is a basis* $v_1, v_2, \dots, v_{2n}$ *of V such that the matrix associated to this form*

$$((v_i, v_j))_{1 \leq i,j \leq 2n} = diag(1, 1, \dots, 1, -1, -1, \dots, -1)$$

Where there are p 1's and q (-1)'s.The difference p-q is called the signature of the form.

2.1.7 Definition : Skew Hermitian Form

Definition 2.7. Let F be a field and let V be a finite dimensional vector space over F. A Skew Hermitian form on V is a function

$$V \times V \ni u, v \to (u, v) \in F$$

such that for all $u, u', v \in V$ and $a \in \mathbb{F}$

$$(u, v) = -\overline{(v, u)}$$
$$(u + u', v) = (u, v) + (u', v)$$
$$(au, v) = a(u, v)$$

2.1.8 Definition : Isometry Group of Bilinear Forms

Definition 2.8. two symmetric forms (,) and (,)' on a finite dimensional vector space V are isometric if and only if there is a linear bijection $Q : V \to V$ such that

$$(Qu, Qv)' = (u, v)(u, v \in V).$$

2.1.9 Fact : Isometry Group of Bilinear Forms

Theorem 2.9. *Let (,) and* $(,)'$ *be two non-degenerate forms on V of the same type, as described above and let* G, G' *denote the corresponding isometry groups. Then if the forms are isometric then the groups are isomorphic. In particular, up to a group isomorphism, the following list exhausts the isometry groups of non-degenerate forms over R or C:*

$O(p,q), Sp_{2n}(\mathbb{R}); On(\mathbb{C}); Sp_2n(\mathbb{C}); U(p,q)$

2.2 Polar (Cartan) Decomposition

Let V be a finite dimensional vector space over $\mathbb{C}$ with a positive definite hermitian form $(\ ,\ )$. For $T \in End(V)$ let $T* \in End(V)$ be defined by

$$(Tu, v) = (u, T * v) \qquad (u, v \in V)$$

An endomorphism $T \in End(V)$ is hermitian if $T = T*$ and it is positive definite if $(Tv; v) > 0$ for all non-zero $v \in V$. Denote by $P \subseteq End(V)$ the subset of all the hermitian positive definite endomorphisms. Let $K = \{k \in GL(V) : k* = k^{-1}\}$ Then $K \times P$ is homeomorphic to $GL(V)$.

Chapter 3

Lie Algebras and the Exponential Mapping

The exponential of a matrix is very important as using the exponential map we pass from Lie algebra to matrix Lie group. It is oftentimes easy to solve a problem at Lie algebra level and then come back to Lie group.

3.1 Definition: Matrix Exponential

Exponential of a matrix X is defined as

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} \tag{3.1}$$

3.1.1 Definition: Operator Norm

Suppose V is a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ and $End(V) \ni T$ then

$$\|T\| = \max\{\|Tv\|_\infty : v \in V \text{ with } \|v\| = 1\}$$

3.1.2 Definition: Matrix Norm / Hilbert-Schmidt Norm

The matrix norm may be defined by considering it as the space $M_n(\mathbb{C})$ of all $n \times n$ matrices as $\mathbb{C}^{n^2}$. For $A = (a_{i,j}) \in M_n(\mathbb{C})$ we have,

$$\|A\| = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2} \tag{3.2}$$

3.1.3 Matrix Norm Properties

$$\|ST\|_\infty \leq \|S\|_\infty \|T\|_\infty \tag{3.3}$$

$$\|T\|_\infty = 0 \iff T = 0 \tag{3.4}$$

Proposition 3.1. *If A_k is a sequence of $n \times n$ real or complex matrices such that A_k is a Cauchy sequence, then there exists a unique matrix A such that A_k converges to A.*

That is, every Cauchy sequence converges.

Now, consider an infinite series whose terms are matrices:

$$A_0 + A_1 + A_2 + \cdots. \tag{3.5}$$

If

$$\sum_{m=0}^{\infty} \|A_m\| < \infty$$

then the series (3.5) is said to **converge absolutely**. If a series converges absolutely, then partial sums of the series form a Cauchy sequence, and hence by Proposition 3.1, the series converges. That is, any series which converges absolutely also converges.

This justifies our definition of Matrix exponential in 3.1 as the series on the right is absolutely convergent.

3.1.4 Fact: Matrix Exponential Properties

Let X, Y be arbitrary $n \times n$ matrices. Then

1. $e^0 = I$.
2. e^X is invertible, and $\left(e^X\right)^{-1} = e^{-X}$.
3. $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ for all real or complex numbers α, β.
4. If $XY = YX$, then $e^{X+Y} = e^X e^Y = e^Y e^X$.
5. If C is invertible, then $e^{CXC^{-1}} = Ce^X C^{-1}$.

6. $\|e^X\| \leq e^{\|X\|}$.

3.2 Computing the Exponential of a Matrix

3.2.1 Case 1: X is diagonalizable

Suppose that X is a $n \times n$ real or complex matrix, and that X is diagonalizable over $\mathbb{C}$, that is, that there exists an invertible complex matrix C such that $X = CDC^{-1}$, with

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Observe that e^D is the diagonal matrix with eigenvalues $e^{\lambda_1}, \cdots, e^{\lambda_n}$, and so in light of Proposition 3.1.4, we have

$$e^X = C \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} C^{-1}.$$

3.2.2 Case 2: X is nilpotent

An $n \times n$ matrix X is said to be **nilpotent** if $X^m = 0$ for some positive integer m. Of course, if $X^m = 0$, then $X^l = 0$ for all $l > m$. In this case the series 3.1 which defines e^X terminates after the first m terms, and so can be computed explicitly.

For example, compute e^{tX}, where

$$X = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$X^2 = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and that $X^3 = 0$. Thus

$$e^{tX} = \begin{pmatrix} 1 & ta & tb + \frac{1}{2}t^2ac \\ 0 & 1 & tc \\ 0 & 0 & 1 \end{pmatrix}.$$

3.2.3 Case 3: X arbitrary

A general matrix X may be neither nilpotent nor diagonalizable. However, it follows from the Jordan canonical form that X can be written in the form $X = S + N$ with S diagonalizable, N nilpotent, and $SN = NS$. Then, since N and S commute,

$$e^X = e^{S+N} = e^S e^N$$

and e^S and e^N can be computed as above.

Theorem 3.2. *Let X be an $n \times n$ real or complex matrix. Then*

$$\det\left(e^X\right) = e^{\mathrm{trace}(X)}.$$

3.3 Campbell Hausdorff Formula

At first we note that $e^X + Y = e^X e^Y$ is true only if X and Y commute.

Definition 3.3. Given two $n \times n$ matrices A and B, the **bracket** (or **commutator**) of A and B is defined to be simply

$$[A, B] = AB - BA.$$

Theorem 3.4. *Campbell Hausdorff Formula: Suppose $A, B, C \in M_n(\mathbb{C})$ are such that $||A||, ||B||, ||C|| \leq \frac{1}{2}$ and $e^X e^Y = e^C$ then*

$$C = A + B + \frac{1}{2}[A, B] + S$$

where

$$||S|| \leq 65(||A|| + ||B||)^3$$

3.3.1 Fact : Some Further Properties of Matrix Exponential

Proposition 3.5. *Let $A, B \in M_n\mathbb{C}$. Then*

$$\lim_{n\to\infty}\left(e^{\frac{A}{n}}e^{\frac{B}{n}}\right)^n = e^{A+B}.$$

Proposition 3.6. *Let $A, B \in M_n\mathbb{C}$. Then*

$$\lim_{n\to\infty}\left(e^{\frac{A}{n}}e^{\frac{B}{n}}\right)^{n^2} = e^{[A,B]}.$$

3.4 Lie Algebra

Definition 3.7. Let G be a matrix Lie group. Then the **Lie algebra** of G, denoted by $\mathfrak{g}$, is the set of all matrices X such that $\forall t \in \mathbb{R}$ we have $e^{tX} \in G$.

3.4.1 The general linear groups

Let $X \in M_n(\mathbb{C})$, then by Proposition 3.1.4, e^{tX} is invertible. Thus the Lie algebra of $\mathsf{GL}(n;\mathbb{C})$ is the space of all $n \times n$ complex matrices. This Lie algebra is denoted $\mathsf{gl}(n;\mathbb{C})$.

Similarly, the Lie algebra of $\mathsf{GL}(n;\mathbb{R})$ is the space of all $n \times n$ real matrices, denoted $\mathsf{gl}(n;\mathbb{R})$.

3.4.2 The special linear groups

We know from Theorem 3.2: $\det\left(e^X\right) = e^{\mathrm{trace}X}$. Hence, if $\mathrm{trace}X = 0$, then $\det\left(e^{tX}\right) = 1$ for all real t. Conversely, $\det\left(e^{tX}\right) = 1$ for all t, $\implies$ $e^{(t)(\mathrm{trace}X)} = 1$ for all t. This means that $(t)(\mathrm{trace}X)$ is an integer multiple of $2\pi i$ for all t, $\implies$ $\mathrm{trace}X = 0$. Thus the Lie algebra of $\mathsf{SL}\,(n;\mathbb{C})$ is the space of all $n \times n$ complex matrices with trace zero, denoted $\mathsf{sl}(n;\mathbb{C})$.

Similarly, the Lie algebra of $\mathsf{SL}\,(n;\mathbb{R})$ is the space of all $n \times n$ real matrices with trace zero, denoted $\mathsf{sl}\,(n;\mathbb{R})$.

3.4.3 The unitary groups

We know that a matrix U is unitary if and only if $U^* = U^{-1}$. Therefore e^{tX} is unitary $\iff$

$$\left(e^{tX}\right)^* = \left(e^{tX}\right)^{-1} = e^{-tX}. \tag{3.6}$$

But by taking adjoints term-by-term, we see that $\left(e^{tX}\right)^* = e^{tX^*}$, and so (3.6) becomes

$$e^{tX^*} = e^{-tX}. \tag{3.7}$$

Clearly, if $X^* = -X$ then (3.7) holds. Conversely, if (3.7) holds for all t, then by differentiating at $t = 0$, we see that $X^* = -X$.

Thus the Lie algebra of $\mathsf{U}(n)$ is the space of all $n \times n$ complex matrices X such that $X^* = -X$, denoted $\mathsf{u}(n)$.

By combining the two previous computations, we see that the Lie algebra of $\mathsf{SU}(n)$ is the space of all $n \times n$ complex matrices X such that $X^* = -X$ and trace$X = 0$, denoted $\mathsf{su}(n)$.

3.4.4 The orthogonal groups

Thus the Lie algebra of $\mathsf{O}(n)$, as well as the Lie algebra of $\mathsf{SO}(n)$, is the space of all $n \times n$ real matrices X with $X^{tr} = -X$, denoted $\mathsf{so}(n)$.

The same argument shows that the Lie algebra of $\mathsf{SO}(n;\mathbb{C})$ is the space of $n \times n$ complex matrices satisfying $X^{tr} = -X$, denoted $\mathsf{so}(n;\mathbb{C})$. This is not the same as $\mathsf{su}(n)$.

3.4.5 The generalized orthogonal groups

A matrix A is in $\mathsf{O}(n;k)$ if and only if $A^{tr} g A = g$, where g is the $(n+k) \times (n+k)$ diagonal matrix with the first n diagonal entries equal to one, and the last k diagonal entries equal to minus one. This condition is equivalent to the condition $g^{-1} A^{tr} g = A^{-1}$, or, since explicitly $g^{-1} = g$, $g A^{tr} g = A^{-1}$. Now, if X is an $(n+k) \times (n+k)$ real matrix, then e^{tX} is in $\mathsf{O}(n;k)$ if and only if

$$g e^{tX^{tr}} g = e^{tgX^{tr}g} = e^{-tX}.$$

This condition holds for all real t if and only if $gX^{tr}g = -X$. Thus the Lie algebra of $\mathsf{O}(n;k)$, which is the same as the Lie algebra of $\mathsf{SO}(n;k)$, consists of all $(n+k) \times (n+k)$ real matrices X with $gX^{tr}g = -X$. This Lie algebra is denoted $\mathsf{so}(n;k)$.

(In general, the group $\mathsf{SO}(n;k)$ will not be connected, in contrast to the group $\mathsf{SO}(n)$. The identity component of $\mathsf{SO}(n;k)$, which is also the identity component of $\mathsf{O}(n;k)$, is denoted $\mathsf{SO}(n;k)_I$. The Lie algebra of $\mathsf{SO}(n;k)_I$ is the same as the Lie algebra of $\mathsf{SO}(n;k)$.)

3.4.6 The symplectic groups

These are denoted $\mathsf{sp}\,(n;\mathbb{R})$, $\mathsf{sp}(n;\mathbb{C})$, and $\mathsf{sp}\,(n)$. The calculation of these Lie algebras is similar to that of the generalized orthogonal groups, and I will just record the result here. Let J be the matrix in the definition of the symplectic groups. Then $\mathsf{sp}\,(n;\mathbb{R})$ is the space of $2n \times 2n$ real matrices X such that $JX^{tr}J = X$, $\mathsf{sp}(n;\mathbb{C})$ is the space of $2n \times 2n$ complex matrices satisfying the same condition, and $\mathsf{sp}\,(n) = \mathsf{sp}(n;\mathbb{C}) \cap \mathsf{u}\,(2n)$.

3.5 Properties of the Lie Algebra

We will now establish various basic properties of the Lie algebra of a matrix Lie group. The reader is invited to verify by direct calculation that these general properties hold for the examples computed in the previous section.

Proposition 3.8. *Let G be a matrix Lie group, and X an element of its Lie algebra. Then e^X is an element of the identity component of G.*

Proof. By definition of the Lie algebra, e^{tX} lies in G for all real t. But as t varies from 0 to 1, e^{tX} is a continuous path connecting the identity to e^X. □

Proposition 3.9. *Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. Let X be an element of $\mathfrak{g}$, and A an element of G. Then AXA^{-1} is in $\mathfrak{g}$.*

Proof. This is immediate, since by Proposition 3.1.4,

$$e^{t(AXA^{-1})} = Ae^{tX}A^{-1},$$

and $Ae^{tX}A^{-1} \in G$. □

Theorem 3.10. *Let G be a matrix Lie group, $\mathfrak{g}$ its Lie algebra, and X, Y elements of $\mathfrak{g}$. Then*

1. *$sX \in \mathfrak{g}$ for all real numbers s,*

2. *$X + Y \in \mathfrak{g}$,*

3. *$XY - YX \in \mathfrak{g}$.*

Theorem 3.11. *Let G and H be matrix Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Suppose that $\phi : G \to H$ be a Lie group homomorphism. Then there exists a unique real linear map $\widetilde{\phi} : \mathfrak{g} \to \mathfrak{h}$ such that*

$$\phi(e^X) = e^{\widetilde{\phi}(X)}$$

for all $X \in \mathfrak{g}$. The map $\widetilde{\phi}$ has following additional properties

1. $\widetilde{\phi}\left(AXA^{-1}\right) = \phi(A)\widetilde{\phi}(X)\phi(A)^{-1}$, *for all $X \in \mathfrak{g}$, $A \in G$.*
2. $\widetilde{\phi}([X, Y]) = \left[\widetilde{\phi}(X), \widetilde{\phi}(Y)\right]$, *for all $X, Y \in \mathfrak{g}$.*
3. $\widetilde{\phi}(X) = \frac{d}{dt}\big|_{t=0} \phi(e^{tX})$, *for all $X \in \mathfrak{g}$.*

3.6 Complexification

Definition 3.12. Let V be a vector space over $\mathbb{R}$ then complexification of V, denoted by $V_{\mathbb{C}}$ is $\mathbb{C} \otimes_{\mathbb{R}} V$. In other words, $V_{\mathbb{C}}$, is the space of formal linear combinations

$$v_1 + iv_2$$

with $v_1, v_2 \in V$. This becomes a real vector space in the obvious way, and becomes a complex vector space if we define

$$i(v_1 + iv_2) = -v_2 + iv_1.$$

Proposition 3.13. *Let U be a finite dimensional vector space over $\mathbb{C}$ and let $\sigma \in \mathrm{End}_{\mathbb{R}}(U)$ be such that $\sigma^2 = I$ and $\sigma(zu) = \bar{z}\sigma(u) \forall z \in \mathbb{C}, u \in U$. Then $U \cong V_{\mathbb{C}}$ where $V = \{u \in U : \sigma(u) = u\}$.*

Chapter 4

Representations

4.1 Representation of Lie Group

Definition 4.1. Let G be a matrix Lie group. Then a **finite-dimensional complex representation** of G is a Lie group homomorphism

$$\Pi : G \to \mathsf{GL}(V)$$

where V is a finite-dimensional complex vector space. A **finite-dimensional real representation** of G is a Lie group homomorphism into $\mathsf{GL}(V)$, where V is a finite-dimensional real vector space.

4.2 Representation of Lie Algebra

Definition 4.2. If $\mathfrak{g}$ is a real or complex Lie algebra, then a **finite-dimensional complex representation** of $\mathfrak{g}$ is a Lie algebra homomorphism π of $\mathfrak{g}$ into $\mathsf{gl}(n; \mathbb{C})$ or into $\mathsf{gl}(V)$, where V is a finite-dimensional complex vector space. If $\mathfrak{g}$ is a *real* Lie algebra, then a **finite-dimensional real representation** of $\mathfrak{g}$ is a Lie algebra homomorphism π of $\mathfrak{g}$ into $\mathsf{gl}(n; \mathbb{R})$ or into $\mathsf{gl}(V)$.

If Π or π is a one-to-one homomorphism, then the representation is called **faithful**.

4.3 Adjoint Representation

Definition 4.3. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$.

$$\mathrm{Ad} : G \to \mathsf{GL}(\mathfrak{g})$$

by the formula

$$\mathsf{Ad}A(X) = AXA^{-1}.$$

It is easy to see that Ad is a Lie group homomorphism. Since Ad is a Lie group homomorphism into a group of invertible operators, we see that in fact Ad is a representation of G, acting on the space $\mathfrak{g}$. This is called **adjoint representation** of G. The adjoint representation is a real representation of G.

Definition 4.4. Let $\mathfrak{g}$ be a Lie algebra, we have

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

defined by the formula

$$\mathsf{ad}X(Y) = [X, Y].$$

Proposition 4.5. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$, and let Π be a (finite-dimensional real or complex) representation of G, acting on the space V. Then there is a unique representation π of $\mathfrak{g}$ acting on the same space such that*

$$\Pi(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$. The representation π can be computed as

$$\pi(X) = \left.\frac{d}{dt}\right|_{t=0} \Pi\left(e^{tX}\right)$$

and satisfies

$$\pi\left(AXA^{-1}\right) = \Pi(A)\pi(X)\Pi(A)^{-1}$$

for all $X \in \mathfrak{g}$ and all $A \in G$.

4.4 Irreducible Representation

Definition 4.6. Let Π be a finite-dimensional real or complex representation of a matrix Lie group G, acting on a space V. A subspace W of V is called **invariant** if $\Pi(A)w \in W$ for all $w \in W$ and all $A \in G$. An invariant subspace W is called **non-trivial** if

$W \neq \{0\}$ and $W \neq V$. A representation with no non-trivial invariant subspaces is called **irreducible**.

The terms **invariant**, **non-trivial**, and **irreducible** are defined analogously for representations of Lie algebras.

4.5 Schur's Lemma

Theorem 4.7. *Let V be a finite dimensional complex vector space and let (π, V) be an irreducible representation of G on V, then*

$$\mathrm{End}_G(V) := \{T \in \mathrm{End}(V) : \pi(g)T = T\pi(g), \quad \forall g \in G\} = \mathbb{C}I$$

4.6 Finite Dimensional Irreducible Representation of $sl_2(\mathbb{C})$

Theorem 4.8. *For each integer $m \geq 0$, there is an irreducible representation of* $\mathsf{sl}(2;\mathbb{C})$ *with dimension $m+1$. Any two irreducible representations of* $\mathsf{sl}(2;\mathbb{C})$ *with the same dimension are equivalent.*

4.7 Unitary Representation of Topological Group

Definition 4.9. A unitary representation of a topological group G is a Hilbert Space V over $\mathbb{C}$ and a group homomorphism $\pi : G \to$ **Unitary Operator**(V) such that the maps $G \ni g \to \pi(g)v \in V$ are continuous for any $v \in V$.

Proposition 4.10. *The only irreducible finite dimensional Unitary representation of $SL_2\mathbb{R}$ is the trivial representation. (Up to equivalence)*

Proposition 4.11. *The only irreducible finite dimensional Unitary representation of $\mathbb{R}$ are one dimensional representation given by, $(V = \mathbb{C})$,*

$$\pi_\zeta(x) = e^{i\zeta x} \quad (\zeta \in \mathbb{R})$$

Theorem 4.12. *Let G be a topological group. If G is abelian then every irreducible unitary representation of G is one-dimensional.*

Chapter 5

Fourier Analysis

Definition 5.1. Let $S(\mathbb{R})$ denote the **Schwartz space** of $\mathbb{R}$. This is the complex vector space of all the smooth functions $f : R \to C$ (i.e. C^∞) such that for any two non-negative integers n, k,

$$\sup_{x\in\mathbb{R}} |x^n \delta^k f(x)| \leq \infty$$

A Schwartz function is clearly integrable.

Theorem 5.2. *Let $\mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R})$ denote the Fourier transform:*

$$\mathcal{F}f(y) = \int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx \quad (y \in \mathbb{R}).$$

Then we have the Fourier Inversion Formula

$$\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}} f(\zeta)e^{2\pi ix\zeta}d\zeta \quad (f \in S(\mathbb{R})).$$

We can extend the above definition of Fourier transforms for any function $f \in L^2(\mathbb{R})$.

Bibliography

[1] B. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer, 2003. ISBN 9780387401225. URL http://books.google.com/books?id=m1VQi8HmEwcC.

[2] Roe Goodman. *Symmetry, representations, and invariants*. Graduate Texts in Mathematics. Springer, 2009. ISBN 978-0-387-79851-6.

[1] [2]