A useful integration formula

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Claim 1: Assume k to be a positive integer such that $\Re(s+k) > 1$, then we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x+i)^s (x-i)^k} = \pi i^{k-s}\, 2^{2-s-k} \frac{\Gamma(s+k-1)}{\Gamma(s)\Gamma(k)}.$$

Proof. Let

$$f(z) = \frac{1}{(z+i)^s (z-i)^k}.$$

Let us consider the contour of integration γ to be a semicircle with center at the origin with radius R. The diameter is from $(-R, 0)$ to $(R, 0)$ along the $x-$axis and the circular arc of radius R from $\theta = 0$ to $\theta = \pi$ in the counterclockwise direction. The only pole enclosed in the interior of γ is at $z = i$. The residue theorem implies that

$$\int_\gamma f(z)\, dz = 2\pi i\, Res(f, i). \tag{1}$$

We have, $\Re(s+k) > 1$ therefore as $R \to \infty$ the contribution of the circular arc from $\theta = 0$ to $\theta = \pi$ in the above integral goes to zero. So, (1) implies that

$$\int_\gamma f(z)\, dz = \int_{-\infty}^{\infty} \frac{dx}{(x+i)^s (x-i)^k} = 2\pi i\, Res(f, i). \tag{2}$$

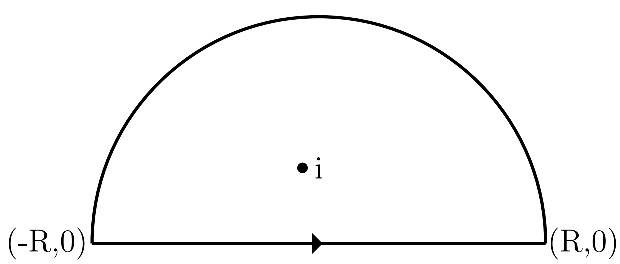

Figure 1:

Since $f(z)$ has a pole of order k at $z = i$, we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x+i)^s (x-i)^k} = 2\pi i \, Res(f,i) = \frac{2\pi i}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[\frac{1}{(z+i)^s} \right]_{z=i}$$
$$= \frac{2\pi i}{(k-1)!} \frac{(-s)(-s-1)\cdots(-s-k+2)}{(2i)^{s+k-1}}$$
$$= \pi i^{k-s} \, 2^{2-s-k} \frac{\Gamma(s+k-1)}{\Gamma(s)\Gamma(k)}$$

□

Claim 2: Assume $\Re(s+t) > 1$, then we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x+i)^s (x-i)^t} = \pi i^{t-s} \, 2^{2-s-t} \frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}.$$

Proof. We note that this formula is due to Cauchy (See (102) on the page 34 of [1]).

$$\int_{-\infty}^{\infty} \frac{dx}{(x+i)^s (x-i)^t} = \int_{-\infty}^{\infty} \frac{i^{t-s} \, dx}{(1-ix)^s (1+ix)^t}. \tag{3}$$

Let

$$f(z) = \frac{1}{(1-iz)^s (1+iz)^t}.$$

Let γ be the contour of integration as shown in the Figure 2. We note that f is holomorphic inside the region enclosed by γ. So we get that,

$$\int_{\gamma} f(z) \, dz = 0.$$

We have $\Re(s+t) > 1$, therefore as $R \to \infty$ the contributions of the big circular arcs are zero. Similarly, as the radius $\epsilon \to 0$ the contribution of the small circular

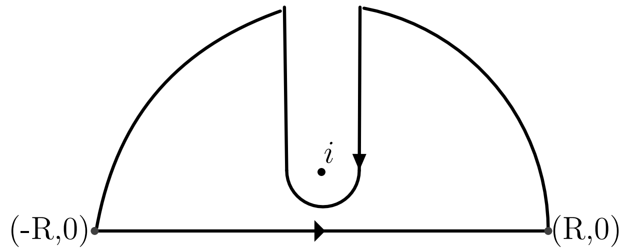

Figure 2:

arc is also zero. Therefore,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{(1-ix)^s(1+ix)^t}\,dx \\
&= \lim_{\epsilon\to 0}\left\{\int_1^{\infty} \frac{i\,dy}{(1+y-i\epsilon)^s(1-y+i\epsilon)^t} - \int_1^{\infty} \frac{i\,dy}{(1+y+i\epsilon)^s(1-y-i\epsilon)^t}\right\} \\
&= \lim_{\epsilon\to 0}\left\{\int_1^{\infty} \frac{(-1)^{-t} i\,dy}{(1+y-i\epsilon)^s(y-1-i\epsilon)^t} - \int_1^{\infty} \frac{(-1)^{t} i\,dy}{(1+y+i\epsilon)^s(y-1+i\epsilon)^t}\right\} \\
&= \int_1^{\infty} \frac{\{(-1)^{-t}-(-1)^t\} i\,dy}{(1+y)^s(y-1)^t} \\
&= 2\sin(t\pi)\int_1^{\infty} \frac{dy}{(1+y)^s(y-1)^t} \\
&= 2\sin(t\pi)\int_0^{\infty} \frac{dy}{y^t(y+2)^s} \\
&= 2\sin(t\pi)\int_0^{1} \frac{2^{1-t-s}\,d\tau}{\tau^t(1-\tau)^{2-s-t}} \qquad \text{(using the substituition } \tau = \frac{y}{y+2}\text{)} \\
&= 2^{2-t-s}\,\frac{\pi}{\Gamma(t)\Gamma(1-t)}\,\mathcal{B}(-t+1, s+t-1) \\
&= \pi\, 2^{2-s-t} \frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}.
\end{aligned}$$

Finally, using (3) we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(x+i)^s(x-i)^t} = \int_{-\infty}^{\infty} \frac{i^{t-s}\,dx}{(1-ix)^s(1+ix)^t} = \pi\, i^{t-s}\, 2^{2-s-t} \frac{\Gamma(s+t-1)}{\Gamma(s)\Gamma(t)}.$$

□

References

[1] Cauchy, A.-L. (1825), Sur les integrales definies prises entre des limites imaginaires, Bulletin de Ferussoc, T. III, 214-221, Oeuvres de Cauchy, 2 e serie, T. II, Gauthier-Villars, Paris, 1958, 57-65.