Calculus 2 - Discussion

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The Sigma Notation

Suppose we want to calculate the sum of consecutive integers from 1 to 100. One way to represent this would be to write $1 + 2 + 3 \ldots 99 + 100$. In this case it is easy to guess the pattern. But, in more complicated cases we would like to have a precise mathematical notation. So, we use the symbol "$\sum$" to express the above sum as $\displaystyle\sum_{i=1}^{100} i$.

Let us work out a few examples. Can you write the following sums in $\sum$ notation?

Q 1. $1 + 3 + 5 + 7 + \ldots 97 + 99$

Q 2. $2 + 4 + 6 + \ldots 98 + 100$

Q 3. $1 + 1 + 1 + \ldots 1$ (up to 100 terms.)

The answers are -

A 1. $\sum_{i=1}^{50}(2 * i - 1) = 1 + 3 + 5 + 7 + \ldots 99$

A 2. $\sum_{i=1}^{50}(2 * i) = 2 + 4 + 6 + \ldots 100$

A 3. $\sum_{i=1}^{100} 1 = 1 + 1 + 1 + \ldots 1$ (up to 100 terms.)

Some Properties

1. $$\sum_{n=s}^{t} C \cdot f(n) = C \cdot \sum_{n=s}^{t} f(n)$$
2. $$\sum_{n=s}^{t} f(n) + \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} [f(n) + g(n)]$$
3. $$\sum_{n=s}^{t} f(n) - \sum_{n=s}^{t} g(n) = \sum_{n=s}^{t} [f(n) - g(n)]$$
4. $$\sum_{n=s}^{t} f(n) = \sum_{n=s+p}^{t+p} f(n-p)$$
5. $$\sum_{n=s}^{j} f(n) + \sum_{n=j+1}^{t} f(n) = \sum_{n=s}^{t} f(n)$$
6. $$\sum_{i=k_0}^{k_1} \sum_{j=l_0}^{l_1} a_{i,j} = \sum_{j=l_0}^{l_1} \sum_{i=k_0}^{k_1} a_{i,j}$$

Arihmetic Progression

An arithmetic progression (AP) or arithmetic sequence is a sequence of numbers such that the difference between the consecutive terms (i.e. the common difference) is constant.
If the first term of AP is a_1 and the common difference is d then the n^{th} term a_n is given by $a_n = a_1 + (n-1)d$.

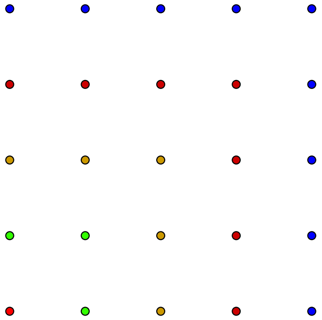
For example, in the series $\sum_{i=1}^{50}(2 * i + 1) = 1 + 3 + 5 + 7 + \ldots 99$,the first term $a_1 = 1$, the common difference is $d = 5 - 3 = 2$ and the n^{th} term a_n is given by $a_n = a_1 + (n-1)d = 1 + (n-1)2 = 2n - 1$.

The Sum of an AP

$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d)$
$S_n = (a_n - (n-1)d) + (a_n - (n-2)d) + \cdots + (a_n - 2d) + (a_n - d) + a_n$.
On adding both sides of the two equations,we get $2S_n = n(a_1 + a_n)$. Thus we get, using $a_n = a_1 + (n-1)d$ $S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}[2a_1 + (n-1)d]$

An Example

We have from the formula above, $\sum_{i=1}^{n}(2 * i - 1) = n^2$.
The picture below explains the above equality for $n = 5$. Note that, we can start counting from the bottom left corner and count the dots having the same color to get the total number of dots as $1 + 3 + 5 + 7 + 9 = 5^2$.



Some Imoprtant Results

- $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
- $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
- $$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Please see example 5 (Appendix E) for the proof of the second identity above. The method of proof of example 5 can be generalized to find the expression for sum of $3^{rd}, 4^{th}$ and higher powers of n consecutive integers.

Telescoping Sum

A sum in which consecutive terms cancel each other, leaving only the first and the final terms. This is a very useful trick, that can be used in many cases, to find the sum of a series.For example,consider the following series.

- $S = \sum_{i=2}^{99} \frac{1}{i(i+1)} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots\ldots \frac{1}{99.100}$

Note that the n^{th} term of the series a_n can be written as $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Now,

$$\begin{aligned} S &= \sum_{i=2}^{99} \frac{1}{i*(i+1)} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \ldots\ldots \frac{1}{99.100} \\ &= (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + \ldots\ldots + (\frac{1}{99} - \frac{1}{100}) \\ &= \frac{1}{2} - \frac{1}{100} \end{aligned}$$

What about $S = \sum_{i=2}^{\infty} \frac{1}{i * (i+1)}$?

Watch out for this mistake when telescoping!

$$0 = \sum_{n=1}^{\infty} 0 = \sum_{n=1}^{\infty} (1-1) = 1 + \sum_{n=1}^{\infty} (-1+1) = 1$$