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DEPARTMENT OF MATHEMATICS

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DEDICATION

This work is dedicated

to

my parents

KAMAL KISHORE SHUKLA

LALITA SHUKLA

for

encouraging me to follow my dreams!

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Abstract

We give a representation theoretic approach to the Klingen lift generalizing the classical construction of Klingen Eisenstein series to arbitrary level for both paramodular and Siegel congruence subgroups.

Moreover, we give a computational algorithm for describing the one-dimensional cusps of the Satake compactifications for the Siegel congruence subgroups in the case of degree two for arbitrary levels. As an application of the results thus obtained, we calculate the co-dimensions of the spaces of cusp forms in the spaces of modular forms of degree two with respect to Siegel congruence subgroups of levels not divisible by 8.

Chapter 1

A summary of the main results

1.1 Introduction

The primary objects of our concern in this thesis are Klingen Eisenstein series in degree two. One goal is to understand the classical Klingen Eisenstein series using representation theoretic methods. Moreover, our investigations of these classical objects lead us to some interesting results on the co-dimensions of cusp forms in the spaces of Siegel modular forms of degree two with respect to Siegel congruence subgroups. We have employed both the classical methods and representation theoretic techniques in this thesis and we now give a context to our work under these two broad headings.

1.1.1 Classical results

One of the most basic questions about the spaces of modular forms is to ask for the dimensions and the co-dimensions of the spaces of cusp forms. For the spaces of Siegel modular forms of degree two with respect to the full modular group $\mathrm{Sp}(4,\mathbb{Z})$ the answers are well known for several decades. However, the answers for the spaces of modular forms with respect to Siegel congruence subgroups are not so clear. Only some special cases have been treated in the literature. Dimensions of the spaces of cusp forms with respect to $\Gamma_0(p)$ have been computed by Hashimoto [9] for weights $k \geq 5$. For $\Gamma_0(2)$, in [10] Ibukiyama gave the structure of the ring of Siegel modular forms of degree 2. Poor and Yuen [27] computed the dimensions of cusp forms for weights $k = 2, 3, 4$ with respect to $\Gamma_0(p)$ in the case of a small prime p. In [29] Poor and Yuen described the one-dimensional and zero-dimensional cusps of the Satake compactifications for the paramodular subgroups in the degree two case and calculated the co-dimensions of cusp forms. More recently, in [3] Böcherer and Ibukiyama have given a formula for calculating the co-dimensions of the spaces of cusp forms in the spaces of modular forms of degree two with respect to Siegel congruence subgroups of square-free levels. In this work we generalize their result and give a formula for the co-dimensions of the spaces of cusp forms in the spaces of modular forms of degree two with respect to Siegel congruence subgroups of level N with $8 \nmid N$. The method used to find the co-dimensions of the spaces of cusp forms makes use of a result from the theory of Satake compactification. The cusp structure of the Satake compactification encodes information about the co-dimensions of cusp forms and works of several authors indicate that it is an important object worth investigating.

1.1.2 Representation theoretic results

Eisenstein series are important and concrete examples of modular forms. Siegel generalized the classical holomorphic Eisenstein series to higher dimensional Siegel spaces. Klingen in a paper published in [13] further generalized Siegel Eisenstein series to define, what is now known as, Klingen Eisenstein series. Klingen Eisenstein series $E^k_{n,r}$ are Siegel modular forms of degree n and weight k which are constructed by a natural lift from a given Siegel cusp form of degree r and weight k. In his seminal work in [13], Klingen gave conditions for the regions of convergence of such series. In the following we will restrict our attention to the degree two case. The degree two case has been well studied, for example, in a series of papers in the 1980s Mizumoto and Kitaoka [21, 18, 22, 11] gave explicit formulas for Fourier coefficients of Klingen Eisenstein series which are eigenforms under the action of Hecke operators.

In contrast to the attention the classical degree two Klingen Eisenstein series have received there is no comparable explicit work available from the automorphic representation theory point of view. Of course, the general theory of Eisenstein series is well established [19, 23], and is well known to experts, but the author is not aware of any explicit representation theoretic construction of Klingen Eisenstein series in the literature. We note that the classical Klingen Eisenstein series were defined for the group $\mathrm{Sp}(4, \mathbb{Z})$ and do not admit any level structure. However, recently there have been attempts to define Klingen Eisenstein series of level N with respect to the Siegel congruence subgroup $\Gamma_0(N)$ for square free N (c. f., [7]). In this work we give a representation theoretic explicit construction of Klingen Eisenstein series with arbitrary level N with respect to both the paramodular and Siegel congruence subgroups.

1.2 Notations

We shall use the following notations throughout this work unless otherwise stated.

(i) In this thesis, **except in Chapter 8**, we realize the group $\mathrm{GSp}(4)$ as

$$\mathrm{GSp}(4) := \{g \in \mathrm{GL}(4) \mid {}^t gJg = \lambda(g)J \text{ for some } \lambda(g) \in \mathrm{GL}(1)\},$$

with $J = \begin{bmatrix} & I_2 \\ -I_2 & \end{bmatrix}$.

Important Note: In Chapter 8, we realize the group $\mathrm{GSp}(4)$ **using**

$J = \left[\begin{smallmatrix} & J_1 \\ -J_1 & \end{smallmatrix}\right]$ **in the definition above with** $J_1 = \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]$**. We note that this symmetric version of** $\mathrm{GSp}(4)$ **is isomorphic to the classical version of** $\mathrm{GSp}(4)$ **and we denote this isomorphism by the map** $\jmath$ **which interchanges the first two rows and the first two columns of any matrix.**

(ii) By $B(\mathbb{Q})$ we will mean the Borel subgroup of $\mathrm{GSp}(4, \mathbb{Q})$ consisting of the matrices of the form $\{ \begin{bmatrix} * & & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{bmatrix} \mid * \in \mathbb{Q}\}$.

(iii) $Q(\mathbb{Q})$ will denote the Klingen parabolic subgroup of $\mathrm{GSp}(4, \mathbb{Q})$ consisting of the matrices of the form $\{ \begin{bmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{bmatrix} \mid * \in \mathbb{Q}\}$.

(iv) $P(\mathbb{Q})$ denotes the Siegel parabolic subgroup of $\mathrm{GSp}(4, \mathbb{Q})$ consisting of the matrices of the form $\{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix} \mid * \in \mathbb{Q}\}$.

(v) We denote by $\mathrm{K}(N)$ the paramodular congruence subgroup defined as

$$\mathrm{K}(N) := \{ \begin{bmatrix} * & *N & * & * \\ * & * & * & *N^{-1} \\ * & *N & * & * \\ *N & *N & *N & * \end{bmatrix} \cap \mathrm{Sp}(4, \mathbb{Q}) \mid * \in \mathbb{Z}\}.$$

(vi) The local paramodular subgroup $\mathrm{K}_p(p^n)$ of level p^n is defined to be

$$\mathrm{K}_p(p^n) := \{k = \begin{bmatrix} * & *p^n & * & * \\ * & * & * & *p^{-n} \\ * & *p^n & * & * \\ *p^n & *p^n & *p^n & * \end{bmatrix} \in \mathrm{GSp}(4, \mathbb{Q}_p) \mid * \in \mathbb{Z}_p, \det(k) \in \mathbb{Z}_p^\times\}.$$

(vii) $\mathbb{H}_n := \{z \in M_n(\mathbb{C}) \,|\, {}^t z = z, \mathrm{Im}\, z > 0\}$ will denote Siegel half space of degree (or genus) n.

(viii) $s_1 := \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}, \quad s_2 := \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}.$

(ix) For the integer i and the prime p we define $\mathrm{L}_i := \begin{bmatrix} 1 & p^i & & \\ & 1 & & \\ & & 1 & \\ & & -p^i & 1 \end{bmatrix}$.

(x) We define for integer N, $\mathrm{L}_N := \begin{bmatrix} 1 & N & & \\ & 1 & & \\ & & 1 & \\ & & -N & 1 \end{bmatrix}$.

(xi) We define the Siegel congruence subgroup of level N as

$$\Gamma_0(N) = \Gamma_0^4(N) := \{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ a & b & * & * \\ c & d & * & * \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{Z}) \mid a, b, c, d \equiv 0 \mod N\}.$$

(xii) We will use $\Gamma_0^2(N) := \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid c \equiv 0 \mod N\}$ to denote the Hecke congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. Sometimes we will also use the symbol $\Gamma_0(N)$ for $\Gamma_0^2(N)$ and by context the meaning will be clear without any confusion.

(xiii) We define the local Siegel congruence subgroup of $\mathrm{GSp}(4, \mathbb{Q}_p)$ of level p^n by

$$\mathrm{Si}(p^n) := \{\alpha = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ a & b & * & * \\ c & d & * & * \end{bmatrix} \in \mathrm{GSp}(4, \mathbb{Z}_p) \mid a, b, c, d \in p^n\mathbb{Z}_p\}.$$

(xiv) Let $\mathrm{K}_{\mathrm{p}}^2(p^{n_p}) := \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}_p) \mid c \in p^{n_p}\mathbb{Z}_p,\ d \in 1 + p^{n_p}\mathbb{Z}_p\}$.

(xv) Let $\Gamma_\infty^2(\mathbb{Z}) := \{\pm \begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} | \, b \in \mathbb{Z}\}$.

(xvi) $\Gamma_\infty(\mathbb{Z}) := Q(\mathbb{Q}) \cap \mathrm{Sp}(4, \mathbb{Z})$ and $\Delta(\mathbb{Z}/N\mathbb{Z}) := \{ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{Z}/N\mathbb{Z})\}$.

(xvii) $\Gamma_\infty(\mathbb{Z}/N\mathbb{Z}) := \{\gamma \mod N \mid \gamma \in \Gamma_\infty(\mathbb{Z})\}$.

(xviii) For $f : \mathbb{H} \to \mathbb{C}$, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R}^+)$ and $z \in \mathbb{H}$, we define

$$(f|_k \, g)(z) = \det(g)^{\frac{k}{2}}(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

(xix) For $Z \in \mathbb{H}_2 := \{z \in M_2(\mathbb{C}) \,|\, {}^t z = z, \operatorname{Im} z > 0\}$, and for any $m = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in$ $\mathrm{Sp}(4, \mathbb{Z})$ we define $m\langle Z \rangle := (AZ + B)(CZ + D)^{-1}$, $j(m, Z) := CZ + D$ and $m\langle Z \rangle^* = \tilde{\tau}$ for $m\langle Z \rangle = \begin{bmatrix} \tilde{\tau} & \tilde{z} \\ \tilde{z} & \tilde{\tau'} \end{bmatrix}$.

(xx) $C_0(N) = \#(\Gamma_0(N)\backslash \mathrm{GSp}(4, \mathbb{Q})/P(\mathbb{Q}))$ = The number of zero-dimensional cusps for $\Gamma_0(N)$.

(xxi) $C_1(N) = \#(\Gamma_0(N)\backslash \mathrm{GSp}(4, \mathbb{Q})/Q(\mathbb{Q}))$ = The number of one-dimensional cusps for $\Gamma_0(N)$.

(xxii) K = The maximal standard compact subgroup of $\mathrm{Sp}(4, \mathbb{R})$.

(xxiii) K_1 = The maximal standard compact subgroup of $\mathrm{GSp}(4, \mathbb{R})$.

(xxiv) Let $\omega_1(q) := \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for $q = \begin{bmatrix} a & & b & * \\ * & * & * & * \\ c & & d & * \\ & & & * \end{bmatrix} \in Q(\mathbb{Q})$ and let $\imath_1$ be the embedding map $\imath_1(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) := \begin{bmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{bmatrix}$ from $\mathrm{SL}(2, \mathbb{Q})$ to $Q(\mathbb{Q})$. For $g \in \mathrm{GSp}(4, \mathbb{Q})$, we define $\Gamma_g := \omega_1(g^{-1}\Gamma_0(\mathrm{N})g \cap Q(\mathbb{Q}))$.

1.3 Main results

1.3.1 Co-dimension of cusp forms

We recall cusps in the degree one case. Let Γ be a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ which acts on the complex upper half plane $\mathbb{H}$ by the usual action. In order to compactify $\Gamma\backslash\mathbb{H}$ we adjoin $\mathbb{Q} \cup \{\infty\}$ to $\mathbb{H}$ to define the extended plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ and take the quotient $X(\Gamma) = \Gamma\backslash\mathbb{H}^*$. Then a cusp of $X(\Gamma)$ is a Γ-equivalence class of points in $\mathbb{Q} \cup \{\infty\}$. As $\mathrm{SL}(2, \mathbb{Z})$ acts transitively on $\mathbb{Q} \cup \{\infty\}$ there is just one cusp of the modular curve $X(1) = \mathrm{SL}(2, \mathbb{Z})\backslash\mathbb{H}^*$. It is well known that cusps of $X(\Gamma_0^2(N))$ correspond to the double coset decompositions of $\Gamma_0^2(N)\backslash \mathrm{SL}(2, \mathbb{Z})/\Gamma_\infty^2(\mathbb{Z})$, for example see Prop. 3.8.5 in [6] or §4.2 in [20].

The theory of Satake compactification is explained in [33]. Section 3 in [29] gives a quick review. In fact similar to the degree one case, the one-dimensional cusps for the Siegel congruence subgroup $\Gamma_0(N)$, in the degree two case, correspond to the double coset decompositions $\Gamma_0(N)\backslash\mathrm{Sp}(4,\mathbb{Z})/\Gamma_\infty(\mathbb{Z})$ and also equivalently to $\Gamma_0(N)\backslash\mathrm{GSp}(4,\mathbb{Q})/Q(\mathbb{Q})$. Similarly the zero-dimensional cusps correspond to the double coset decompositions $\Gamma_0(N)\backslash\mathrm{GSp}(4,\mathbb{Q})/P(\mathbb{Q})$. It turns out that for even weights $k > 4$, the co-dimension of cusp forms can be obtained by using Satake's theorem (cf. [34]) if the structure of zero-dimensional cusps and one-dimensional cusps are known.

We prove the following result concerning one-dimensional cusps in the case when $N = p^n$ for some prime p and $n \geq 1$. In fact, the one-dimensional cusps for $\Gamma_0(p^n)$ are inverses of the representatives listed below.

Important note: We use the symmetric version of $\mathrm{GSp}(4)$ in Theorem 8.1.10 and Theorem 8.2.2 (c.f. Section 1.2, Notations).

Theorem 8.1.10. ***(Double coset decomposition)***. *Assume $n \geq 1$. A complete and minimal system of representatives for the double cosets $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p^n)$ is given by*

$$1, \qquad s_1 s_2, \qquad g_1(p,\gamma,r) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \gamma p^r & & & 1 \end{bmatrix},\ 1 \leq r \leq n-1,$$

$$g_2(p,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix},\ 1 \leq s \leq n-1,$$

$$g_3(p,\delta,r,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ \delta p^r & p^s & & 1 \end{bmatrix},\ 1 \leq s, r \leq n-1,\ s < r < 2s,$$

where γ, δ runs through elements in $(\mathbb{Z}/f_1\mathbb{Z})^\times$ and $(\mathbb{Z}/f_2\mathbb{Z})^\times$ respectively with $f_1 = \min(r, n-r)$ and $f_2 = \min(2s-r, n-r)$. The total number of representatives given above is

$$C_1(p^n) = \begin{cases} \dfrac{p^{\frac{n}{2}+1} + p^{\frac{n}{2}} - 2}{p-1} & \text{if } n \text{ is even,} \\ \dfrac{2(p^{\frac{n+1}{2}} - 1)}{p-1} & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

The above result can be extended by using the strong approximation theorem and the Chinese remainder theorem to arbitrary N. We have the following lemma.

Lemma 8.2.1. *Assume $N = \prod_{i=1}^{m} p_i^{n_i}$. Then, the number of inequivalent representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(\mathrm{N})$ is given by $C_1(N) = \prod_{i=1}^m C_1(p_i^{n_i})$.*

We have the following result based on Theorem 8.1.10 and Lemma 8.2.1.

Theorem 8.2.2. *Assume $N = \prod_{i=1}^{m} p_i^{n_i}$. A complete and minimal system of representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(N)$ is given by*

$$g_1(\gamma, x) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x\gamma & & & 1 \end{bmatrix}, \ 1 \le \gamma \le N, \ \gamma | N,$$

$$g_3(\gamma, \delta, y) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ \delta & & 1 & \\ y\gamma & \delta & & 1 \end{bmatrix}, \ 1 < \delta < \gamma \le N, \ \gamma|N, \ \delta|N, \ \delta|\gamma, \ \gamma|\delta^2;$$

where for fixed γ and δ we have

$$x = M + \zeta \prod_{p_i \nmid M,\, p_i | N} p_i^{n_i}, \ y = L + \theta \prod_{p_i \nmid L,\, p_i | N} p_i^{n_i}$$

with $M = \gcd(\gamma, \frac{N}{\gamma})$, $L = \gcd(\frac{\delta^2}{\gamma}, \frac{N}{\gamma})$, ζ and θ varies through all the elements of $(\mathbb{Z}/M\mathbb{Z})^\times$ and $(\mathbb{Z}/L\mathbb{Z})^\times$ respectively. Here we interpret $(\mathbb{Z}/\mathbb{Z})^\times$ as an empty set.

We note that the one-dimensional cusps for $\Gamma_0(N)$ are given by the inverses of the representatives listed above.

The number of zero-dimensional cusps $C_0(p^n)$ for odd prime p was calculated by Markus Klein in his thesis (cf. Korollar 2.28 [12]) as

$$C_0(p^n) = 2n + 1 + 2\left(\sum_{j=1}^{n-1} \phi(p^{\min(j,n-j)}) + \sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1} \phi(p^{\min(j,n-i)})\right). \tag{1.2}$$

It is the same as

$$C_0(p^n) = \begin{cases} 3 & \text{if } n = 1, \\ 2p+3 & \text{if } n = 2, \\ -2n - 1 + 2p^{\frac{n}{2}} + 8\frac{p^{\frac{n}{2}}-1}{p-1} & \text{if } n \geq 4 \text{ is even}, \\ -2n - 1 + 6p^{\frac{n-1}{2}} + 8\frac{p^{\frac{n-1}{2}}-1}{p-1} & \text{if } n \geq 3 \text{ is odd}. \end{cases} \tag{1.3}$$

The above formula remains valid if $p = 2$ and $n = 1$. The above result also remains true for $p = 2$ and $n = 2$ as calculated by Tsushima (cf. [38]). Hence, assume $8 \nmid N$ and if $N = \prod_{i=1}^{m} p_i^{n_i}$ then following an argument similar to the one given in the proof of Lemma 8.2.1 we obtain

$$C_0(N) = \prod_{i=1}^{m} C_0(p_i^{n_i}). \tag{1.4}$$

Finally, by using Satake's theorem (cf. [34]) and the formula for $C_0(N)$ and $C_1(N)$ described above we obtain the following dimension formula.

Theorem 9.4.1. *Let $N \geq 1$, $8 \nmid N$ and $k \geq 6$, even, then*

$$\begin{aligned} \dim M_k(\Gamma_0(N)) - \dim S_k(\Gamma_0(N)) = C_0(N) + \left(\sum_{\gamma|N} \phi(\gcd(\gamma, \frac{N}{\gamma}))\right) \dim S_k(\Gamma_0^2(N)) \\ + \sum_{1<\delta<\gamma,\, \gamma|N,\, \delta|\gamma,\, \gamma|\delta^2} {\sum}' \dim S_k(\Gamma_g), \end{aligned} \tag{1.5}$$

where $C_0(N)$ *is given by* (1.4) *if* $N > 1$, $C_0(1) = 1$, ϕ *denotes Euler's totient function, and for a fixed* γ *and* δ *the summation* $\sum'$ *is carried out such that* g *runs through every one-dimensional cusps of the form* $g_3(\gamma, \delta, y)$, *with* y *taking all possible values as in Theorem 8.2.2, and* Γ_g *denotes* $\omega_1(g^{-1}\Gamma_0(\mathrm{N})g \cap Q(\mathbb{Q}))$.

Some remarks.

(i) We note that Markus Klein did not consider the case $4|N$ for calculating the number of zero-dimensional cusps in his thesis. Tsushima provided the result for $N = 4$. Since we refer to their results for the number of zero-dimensional cusps we have this restriction in our theorem. We hope to return to this case in the future.

(ii) The above result in the special case of square-free N reduces to the dimension formula given in [3] for even $k \geq 6$. [3] also treats the case $k = 4$ for square-free N.

Next we note the following theorem which describes a linearly independent set of Klingen Eisenstein series with respect to $\Gamma_0(N)$.

Theorem 9.3.1 . *Assume* $N \geq 1$. *Let* $g_1(\gamma, x)$ *and* $g_3(\gamma, \delta, y)$ *be as in Theorem 8.2.2.*

1. *Let* f_1 *be an elliptic cusp form of even weight* k *with* $k \geq 6$ *and level* N. *Let* g *be a one-dimensional cusp for* $\Gamma_0(N)$ *of the form* $\jmath(g_1(\gamma, x)^{-1})$. *Then*

$$E_g(Z) = \sum_{\xi \in (gQ(\mathbb{Q})g^{-1} \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_1(g^{-1}\xi\langle Z\rangle^*) \det(j(g^{-1}\xi, Z))^{-k},$$

defines a Klingen Eisenstein series of level N *with respect to the Siegel congruence subgroup* $\Gamma_0(N)$.

2. Let f_2 be an elliptic cusp form of even weight k with $k \geq 6$ and level δ. Let h be a one-dimensional cusp for $\Gamma_0(N)$ of the form $\jmath(g_3(\gamma, \delta, y)^{-1})$. Then

$$E_h(Z) = \sum_{\xi \in (hQ(\mathbb{Q})h^{-1} \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_2(h^{-1}\xi\langle Z \rangle^*) \det(j(h^{-1}\xi, Z))^{-k},$$

defines a Klingen Eisenstein series of level N with respect to the Siegel congruence subgroup $\Gamma_0(N)$.

As g and h run through all one-dimensional cusps of the form $\jmath(g_1(\gamma, x)^{-1})$ and $\jmath(g_3(\gamma, \delta, y)^{-1})$ respectively, and for some fixed g and h, as f_1 and f_2 run through a basis of $S_k(\Gamma_0(N))$ and $S_k(\Gamma_0(\delta))$ respectively, the Klingen Eisenstein series thus obtained are linearly independent.

1.3.2 Paramodular Klingen lift

As noted earlier the classical Klingen Eisenstein series as defined by Klingen does not admit any level structure. The representation theoretic formulations of Klingen Eisenstein series have the features that there are no restrictions on the level N with respect to both the paramodular and Siegel congruence subgroups. Also in the case of a paramodular lift, the cusp form appearing in (2.11) can be twisted by a Dirichlet character χ.

In the case of paramodular lift we have the following result.

Theorem 6.2.1. *(**Paramodular Klingen lift**). Let S be a finite set of primes and $N = \prod_{p \in S} p^{n_p}$ be a positive integer. Assume χ to be a Dirichlet character modulo N. Let f be an elliptic cusp form of level N, weight k, with $k \geq 6$ an even integer, and character χ, i.e., $f \in S_k^1(\Gamma_0(N), \chi)$. We also assume f to be a newform. Let ϕ be the automorphic form associated with f and let (π, V_π) be the irreducible cuspidal automorphic representation of* $\mathrm{GL}(2, \mathbb{A})$ *generated by ϕ. Moreover, χ could be viewed* *as a continuous character of ideles, which we also denote by χ. Then there exists a global distinguished vector Φ in the global induced automorphic representation $\chi^{-1}|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$ of $\mathrm{GSp}(4,\mathbb{A})$, for $s = k-2$, such that,*

$$\bar{E}(Z) := \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} (\Phi(\gamma b_Z))(1), \tag{1.6}$$

$$(\textit{with } Z = X + iY \in \mathbb{H}_2, b_Z \in B(\mathbb{R}) \textit{ such that } b_Z\langle[^{i}\ {}_{i}]\rangle = Z),$$

defines an Eisenstein series which is the same as the Klingen Eisenstein series of level N^2 with respect to the paramodular subgroup, defined by

$$E(Z) := \sum_{\gamma \in D(N)\backslash \mathrm{K}(N^2)} f(\mathrm{L}_N\gamma\langle Z\rangle^*) \det(j(\mathrm{L}_N\gamma, Z))^{-k}, \tag{1.7}$$

where,

$$\mathrm{L}_N = \begin{bmatrix} 1 & N & & \\ & 1 & & \\ & & 1 & \\ & & -N & 1 \end{bmatrix} \textit{ and } D(N) = (L_N^{-1}\, Q(\mathbb{Q})\, L_N) \cap \mathrm{K}(N^2).$$

It is to be noted that if one starts from an elliptic newform f of level N then the level of the paramodular Klingen Eisenstein series obtained is N^2. Another interesting feature of this construction is that the level of the paramodular Klingen Eisenstein series does not depend upon the character χ of f.

1.3.3 Siegel congruence Klingen lift

A similar construction for the congruence subgroup $\Gamma_0^4(N)$ turns out to be more difficult than the paramodular case as the relation between the local double coset decomposition $Q(\mathbb{Q}_p)\backslash \mathrm{GSp}(4,\mathbb{Q}_p)/\mathrm{Si}(p^n)$ and the global double coset decomposition $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0^4(p^n)$ is not as straightforward as was the case for the corresponding double coset decompositions involving the paramodular subgroup. For example, the number of double cosets depends on the prime p in the former case while it does not depend on the prime p in the latter case involving the paramodular subgroup. But, with the above result in place, by appropriately selecting the local distinguished vectors and gluing them together one gets a global distinguished vector, which is then used to establish the following theorem.

Theorem 7.3.1. *(**Klingen lift for $\Gamma_0^4(\mathbf{N})$**). Let S be a finite set of primes and $N = \prod_{p\in S} p^{n_p}$ be a positive integer. Let f be an elliptic cusp form of level N, weight k, with $k \geq 6$ an even integer, i.e., $f \in S_k^1(\Gamma_0^2(N))$. We also assume f to be a newform. Let ϕ be the automorphic form associated with f and let (π, V_π) be the irreducible cuspidal automorphic representation of* $\mathrm{GL}(2,\mathbb{A})$ *generated by ϕ. Then there exists a global distinguished vector Φ in the global induced automorphic representation $|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$ of* $\mathrm{GSp}(4,\mathbb{A})$*, for $s = k-2$, such that,*

$$\bar{E}(Z) := \det(Y)^{-k/2} \sum_{\gamma\in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} (\Phi(\gamma b_Z))(1), \tag{1.8}$$

$$(\textit{with } Z = X + iY \in \mathbb{H}_2, b_Z \in B(\mathbb{R}) \textit{ such that } b_Z\langle[\begin{smallmatrix} i & \\ & i\end{smallmatrix}]\rangle = Z),$$

defines an Eisenstein series which is the same as the Klingen Eisenstein series of level N with respect to $\Gamma_0^4(N)$, defined by

$$E(Z) := \sum_{\gamma\in\left(Q(\mathbb{Q})\cap\Gamma_0^4(N)\right)\backslash\,\Gamma_0^4(N)} f(\gamma\langle Z\rangle^*)\det(j(\gamma, Z))^{-k}. \tag{1.9}$$

Next we state the following theorem obtained by picking a different support for the non-archimedean distinguished vectors.

Theorem 7.3.2. *(**Another Klingen lift for $\Gamma_0^4(\mathbf{N})$**). Let S be a finite set of primes and $N = \prod_{p\in S} p^{n_p}$ be a positive integer. Let f be an elliptic cusp form of* *level N, weight k, i.e., $f \in S_k^1(\Gamma_0^2(N))$. We also assume f to be a newform. Let ϕ be the automorphic form associated with f and let (π, V_π) be the irreducible cuspidal automorphic representation of* $\mathrm{GL}(2,\mathbb{A})$ *generated by ϕ. Then there exists a global distinguished vector Φ in the global induced automorphic representation $|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}} \pi$ of* $\mathrm{GSp}(4,\mathbb{A})$, *for $s = k-2$, such that,*

$$\bar{E}(Z) := \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q}) \backslash \mathrm{GSp}(4,\mathbb{Q})} (\Phi(\gamma b_Z))(1), \tag{1.10}$$

$$(\textit{with } Z = X + iY \in \mathbb{H}_2, b_Z \in B(\mathbb{R}) \textit{such that } b_Z \langle [\begin{smallmatrix} i & \\ & i \end{smallmatrix}] \rangle = Z),$$

defines an Eisenstein series which is same as the Klingen Eisenstein series of level N with respect to $\Gamma_0^4(N)$, defined by

$$\begin{aligned} E(Z) &= \sum_{\gamma \in \left((s_1 s_2)^{-1} Q(\mathbb{Q}) s_1 s_2 \cap \Gamma_0^4(N) \right) \backslash \Gamma_0^4(N)} f(s_1 s_2 \gamma \langle Z \rangle^*) \det(j(s_1 s_2 \gamma, Z))^{-k} \\ &= \sum_{\gamma \in \left\{ \left[\begin{smallmatrix} & & & -1 \\ a & & b & \\ & 1 & & \\ c & & d & \end{smallmatrix} \right] \left[\begin{smallmatrix} -\mu & 1 & & \\ -k & l & 1 & \mu \\ & l & & 1 \\ -1 & & & \end{smallmatrix} \right] \cap \mathrm{Sp}(4,\mathbb{R}) \,|\, c,l,k \in N\mathbb{Z},\, a,b,d,\mu \in \mathbb{Z} \right\} \backslash \Gamma_0^4(N)} f(s_1 s_2 \gamma \langle Z \rangle^*) \det(j(s_1 s_2 \gamma, Z))^{-k} \end{aligned} \tag{1.11}$$

Some remarks

(i) In Theorem 7.3.1, a Klingen Eisenstein series of level N, with respect to the group $\Gamma_0^4(N)$, was obtained by using an elliptic cusp form f of level N, weight k, i.e., $f \in S_k^1(\Gamma_0^2(N))$. However, for the paramodular lift, $f \in S_k^1(\Gamma_0^2(N))$ yields a Klingen Eisenstein Series of level N^2 with respect to the group $\mathrm{K}(N^2)$.

(ii) The following table gives a short summary.

Subgroup	f = modular form with respect to	Character	Klingen Eisenstein series with respect to
Paramodular	$\Gamma_0^2(N)$	$\chi \mod N$	$\mathrm{K}(N^2)$
Siegel	$\Gamma_0^2(N)$	1	$\Gamma_0^4(N)$

Table 1.1: Paramodular and Siegel lifts.

1.3.4 A connection with the paramodular conjecture

We recall the paramodular conjecture, which predicts a connection between abelian surfaces and Siegel modular forms of degree 2, just as elliptic curves of conductor N are associated with modular cusp forms of weight 2 and level N with respect to the congruence subgroup $\Gamma_0^2(N)$. Essentially, the paramodular conjecture which is due to Brumer and Kramer [4], proposes that abelian surfaces of conductor N correspond to Siegel modular forms with respect to the paramodular subgroup $\mathrm{K}(N)$. There is some computational evidence to support this conjecture. For instance, in 2015 Cris Poor and David Yuen classified Siegel modular cusp forms of weight two for the paramodular group $\mathrm{K}(p)$ for primes $p < 600$ and found it consistent with the paramodular conjecture [28]. The paramodular conjecture can be reformulated in terms of automorphic representations (see Figure 1.1). We review this briefly in the following.

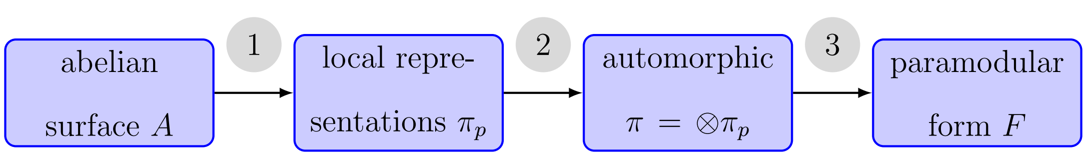

Figure 1.1: Paramodular conjecture reformulated in terms of automorphic representations.

Given any abelian surface of conductor N, the step 1 is to find local representations π_p for each prime p. Let ℓ be a prime different than p and also coprime to N. Then one considers the action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q})$ on the Tate module $T_\ell(A)$ and on $T_\ell(A) \otimes \mathbb{Q}_\ell$. This results in an ℓ-adic representation $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}) \to \mathrm{GL}(4, \mathbb{Q}_\ell)$. Then following the procedure described in §4 of [31], this ℓ-adic representation can be converted to a complex representation σ_p of the Weil-Deligne group $W'(\bar{\mathbb{Q}}_p/\mathbb{Q})$. The determinant of σ_p can be made to be 1 after a twist. Moreover, using the symplectic structure on the Tate module that comes from the Weil pairing, σ_p can be assumed to be a map

$$\sigma_p : W'(\bar{\mathbb{Q}}_p/\mathbb{Q}) \longrightarrow \mathrm{Sp}(4, \mathbb{C}). \tag{1.12}$$

Since the dual group of the split orthogonal group $\mathrm{SO}(5, \mathbb{Q}_p)$ is $\mathrm{Sp}(4, \mathbb{C})$ and the local Langlands correspondence for SO(5) is known, one can now invoke it to associate to σ_p an irreducible, admissible representation π_p of $\mathrm{SO}(5, \mathbb{Q}_p)$. One can also obtain the local archimedean representation π_∞. Hence, in the step 2 combining all these local representations and using the Tensor Product Theorem, one obtains a global representation $\pi = \otimes \pi_p$ of the adelic group $\mathrm{SO}(5, \mathbb{A})$. Since $\mathrm{SO}(5, \mathbb{A}) \cong \mathrm{PGSp}(4, \mathbb{A})$, the paramodular conjecture, in terms of automorphic representations can be formulated as: for an abelian surface A one can associate a global L-packet of representations of $\mathrm{GSp}(4, \mathbb{A})$ with trivial central character such that at least one representation in the L-packet is automorphic. One degenerate case of this conjecture is when the abelian surface is a product of two isogenous elliptic curves, say $E_1 \times E_2$. Interestingly, using the representation theoretic formulation of the paramodular conjecture described above, it can be shown that the distinguished vector Φ in the automorphic representation constructed in Theorem 6.2.1, in the special case of $s = 0$ or equivalently $k = 2$, and $\chi = 1$, corresponds to the abelian surface $E_1 \times E_2$. This confirms and provides a proof of the paramodular conjecture in the special degenerate case of the product of two isogenous elliptic curves.

1.4 Method of proofs

1.4.1 Automorphic representations

The key idea of the proof is to establish the existence of a **global distinguished vector** which corresponds to the classical Klingen Eisenstein series. More explicitly, suppose f is the cusp form appearing in (2.11) and ϕ is the automorphic form associated with f. Let (π, V_π) be the irreducible cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A})$ generated by ϕ. Then, for some appropriately chosen $s \in \mathbb{C}$, there exists a vector Φ in a certain model of the global induced representation $\Pi = |\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}} \pi$ of $\mathrm{GSp}(4, \mathbb{A})$ that corresponds to the Klingen Eisenstein series defined in (2.11).

The global representation Π decomposes as the restricted tensor product of local representations $\Pi = \bigotimes_{p \leq \infty} \Pi_p$. Similarly, the global automorphic representation π associated with the modular form f also breaks up into local constituents $\pi = \otimes \pi_p$. By construction the global representation Π depends upon π and the local components Π_p depend on π_p, just as the classical Klingen Eisenstein series depends upon the modular form f by definition. The non-archimedean local paramodular newform theory established by Brooks Roberts and Ralf Schmidt (see [30]) contains all the necessary ingredients for picking up appropriate local distinguished vectors. Hence by appropriately picking up local distinguished vectors and then gluing then together via restricted tensor products one can create different distinguished vectors in Π. More explicitly, we pick a local newform for each prime $p|N$ which coupled with the choice of certain lowest weight vector for the local archimedean distinguished vector, provides a global distinguished vector, say Ψ, such that Ψ corresponds to a Klingen Eisenstein series with level structure with respect to the paramodular

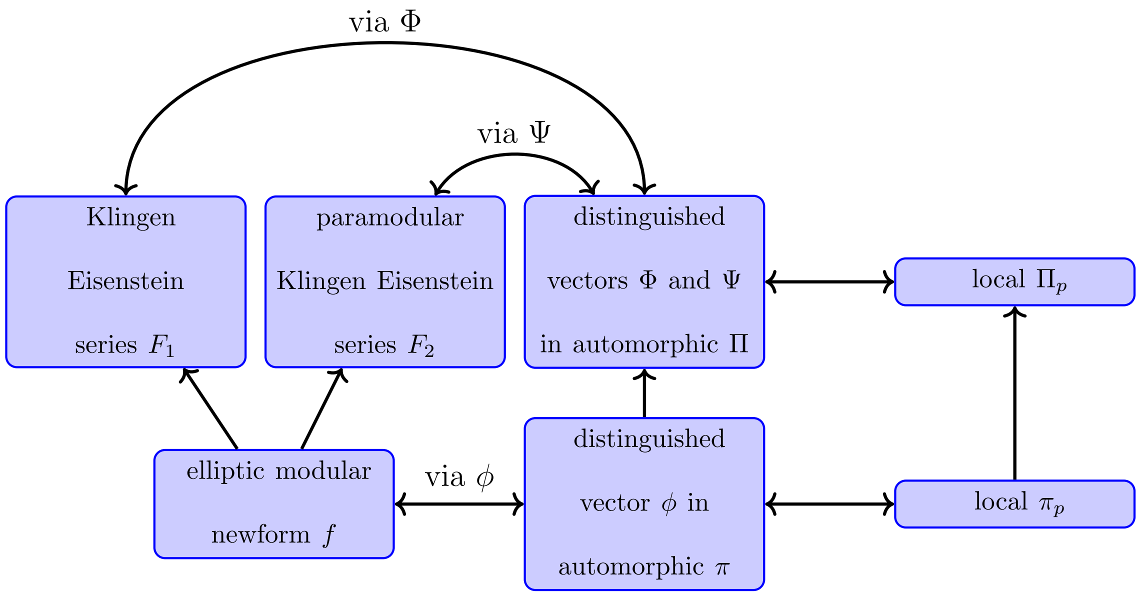

Figure 1.2: Proof sketch for the paramodular Klingen lift.

subgroup (see Figure 1.2).

1.4.2 Co-dimension formula for cusp forms

We noted earlier in Section 1.3.1 that, in the degree 2 case, the one-dimensional cusps for the Siegel congruence subgroup $\Gamma_0(N)$ correspond to the double coset decompositions $\Gamma_0(N)\backslash\mathrm{GSp}(4,\mathbb{Q})/Q(\mathbb{Q})$ and the zero-dimensional cusps correspond to the double coset decompositions $\Gamma_0(N)\backslash\mathrm{GSp}(4,\mathbb{Q})/P(\mathbb{Q})$. Once the structures of zero-dimensional and one-dimensional cusps are known, our co-dimension formula for cusp forms follows from Satake's theorem (cf. [34]).

1.4.2.1 Determining the structure of one-dimensional cusps

It is clear from the earlier discussions that determining the complete structure of one-dimensional cusps is essential for obtaining the main results in this work. We have used Bruhat decomposition and elementary number theory to establish this result. One feature of our method is that it is completely constructive and algorithmic and one can write a computer program to implement it. We have used the computer algebra system ‘Sagemath’ to code and test our algorithm. One downside of our approach is that the proof is rather long. We have restricted ourself to degree 2 but it would be interesting to generalize this result to higher degrees.

A Remark: We note that most of the material in Chapter 8 and Chapter 9 has already appeared in [37].

Chapter 2

A brief historical introduction to modular forms

In this chapter, we begin with a brief historical overview of modular forms. The material is classic and can be skipped by experts.

2.1 Modular forms

Modular forms are very beautiful and special objects which miraculously connect seemingly diverse and disjoint subfields of mathematics, and undoubtedly they have played a significant role in mathematics since they appeared in works of Jacobi, Eisenstein, Fuchs, Dedekind, Klein and Poincaré , to name but a few pioneers, in the nineteenth century mathematics. [1]

Gray, Jeremy gives a very engaging historical account in [8] and describes the

[1] "For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions. I was then very ignorant; every day I seated myself at my work table, stayed an hour or two, tried a great number of combinations and reached no results. One evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. By the next morning I had established the existence of a class of Fuchsian functions, those which come from the hypergeometric series; I had only to write out the results, which took but a few hours."- Poincaré

origin of elliptic and modular functions and their close connections with certain linear differential equations. In the following we give a little glimpse of some of these fascinating connections.

2.1.1 Gauss, AM & GM, elliptic integrals and a modular connection

Gauss' remarkable computational abilities and legendary skills in manipulating infinite series led him to a certain hypergeometric function and its associated second order hypergeometric differential equation. Moreover this hypergeometric function is also closely connected with a certain elliptic integral and therefore to elliptic curves and modular forms. It is instructive to follow his original arguments.

Gauss discovered the arithmetico-geometric mean (agm) when he was 15. He started with two numbers a and b and wrote $a_1 = \dfrac{a+b}{2}$ for the arithmetic mean and $b_1 = \sqrt{ab}$ for the geometric mean of a and b. He then created sequences $\{a_n\}$ and $\{b_n\}$ of arithmetic and geometric means by defining $a_n = \dfrac{a_{n-1}+b_{n-1}}{2}$ and $b_n = \sqrt{a_{n-1}b_{n-1}}$ for $n \geq 1$ with $a_0 = a$ and $b_0 = b$. It is not hard to see that the sequences $\{a_n\}$ and $\{b_n\}$ converge to a common limit, known as the agm of a and b, which Gauss denoted by $M(a,b)$. It is clear that $M(\alpha a, \alpha b) = \alpha M(a,b)$. It is also obvious that $M(1+x, 1-x)$ is an even function. Gauss assumed that its reciprocal has an infinite series expansion.

$$\frac{1}{M(1+x,1-x)} = \sum_{k=0}^{\infty} A_k x^{2k}. \tag{2.1}$$

Now, the substitution $x = \dfrac{2t}{1+t^2}$ leads to

$$\frac{1}{M(1+x,1-x)} = \frac{1+t^2}{M((1+t)^2,(1-t)^2)} = \frac{1+t^2}{M(1+t^2,1-t^2)}$$

giving the relation

$$\sum_{k=0}^{\infty} A_k \left(\frac{2t}{1+t^2}\right)^{2k} = \sum_{k=0}^{\infty} A_k t^{4k}.$$

Gauss determined the coefficients A_k from the above relation and thus he obtained the following infinite series development for $M(1+x, 1-x)^{-1}$

$$y := M(1+x, 1-x)^{-1} = 1 + \left(\frac{1}{2}\right)^2 x^2 + \left(\frac{1.3}{2.4}\right)^2 x^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 x^6 + \cdots \quad . \tag{2.2}$$

Further it satisfies the following differential equation

$$(x^3 - x)\frac{d^2y}{dx^2} + (3x^2 - 1)\frac{dy}{dx} + xy = 0. \tag{2.3}$$

Gauss then noticed a remarkable connection between $M(1+x, 1-x)$ and the complete elliptic integral of the first kind defined as

$$K(x) := \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - x^2 \sin^2 \phi}}\, d\phi. \tag{2.4}$$

Theorem 2.1.1. ***(Gauss).*** *Assume* $|x| < 1$. *Then*

$$K(x) = \frac{\pi}{2} \frac{1}{M(1+x, 1-x)}.$$

Proof. The result follows by expanding $(1 - x^2 \sin^2 \phi)^{-\frac{1}{2}}$ using the binomial theorem, integrating the resulting series term by term and then making use of (2.2). □

It is already amazing to see the creation of a mathematical theory with beautiful interconnections from a seemingly innocent looking idea of agm. However, this was just the beginning. In fact, the general hypergeometric series is defined as

$$F(\alpha, \beta, \gamma, x) := 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots \quad , \tag{2.5}$$

and it is a solution of the general hypergeometric differential equation defined by Gauss as

$$x(1-x)\,\frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)x]\,\frac{dy}{dx} - \alpha\beta y = 0. \tag{2.6}$$

It is readily seen that on substituting $z = x^2$ the equation (2.3) reduces to a special case of the general hypergeometric differential equation defined by (2.6) with $\alpha = \beta = \frac{1}{2}$ and $\gamma = 1$.

The function $M(1,x)$ is also connected with the well known Jacobi Theta Function. We define for $q = e^{2\pi i z}$ with $\mathrm{Im}(z) > 0$,

$$\theta_M(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$
$$\theta_F(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$$

$\theta_M(q)$ is a generating function for the number of ways of representing a number as a sum of squares. More, precisely,

$$\theta_M^k(q) = \sum_{n\geq 0} r_k(n) q^n$$

where,

$$r_k(n) = \#\{(x_1, \cdots x_k) \in \mathbb{Z}^k \mid x_1^2 + \cdots x_k^2 = n\}.$$

It can be shown that

$$\frac{\theta_M^2(q) + \theta_F^2(q)}{2} = \theta_M^2(q^2) \text{ and } \sqrt{\theta_M^2(q)\theta_F^2(q)} = \theta_F^2(q^2), \tag{2.7}$$

from which it follows that

$$M(\theta_M^2(q), \theta_F^2(q)) = 1. \tag{2.8}$$

One common underlying theme in the development of modern mathematics is solution of equations: polynomial equations and the symmetry of their roots resulted in Galois theory, a similar study for the roots of differential equations led to the development of Lie theory and the beautiful subject of algebraic geometry also has its roots in the study of roots of equations! The study of hypergeometric differential equation motivated and provided an impetus for development of a large body of important mathematics such as the theory of complex analysis and also of Riemann surfaces. The solutions of hypergeometric differential equations are messy multivalued functions. New methods and concepts such as that of analytic continuation were invented to systematically study and tame these beasts!

Let us now return to the elliptic integral $K(x)$. These integrals arose naturally in several ways, for example in calculating the time period of a simple pendulum and also in calculating the arc length of an arbitrary ellipse. Legendre studied these integrals extensively and Jacobi and Abel are credited for coming up with the idea of studying the inverse functions of the elliptic integrals instead. The entries in Gauss' personal diaries reveal that he had already discovered most of Jacobi's and Abel's results but chose not to publish them. The other important idea was to study the complex valued functions instead of real valued functions. In this situation we can draw a parallel with the familiar circular trigonometric functions as follows. A circle with equation $y^2 = 1 - x^2$ is parametrized by $x = \sin u$ and $y = \sin' u = \cos u$ and $u = \sin^{-1} x$ is given by the familiar integral $\int_0^x \dfrac{1}{\sqrt{1-t^2}}\, dt$. On extending to the complex domain the $\sin^{-1} x$ function defined by the above integral is a multivalued function. For example one can choose a path for integration from 0 to x on the complex plane such that it goes around an arbitrary number of times around the singularities ± 1. However, if we study the inverse of the function defined by the above integral we get a nice single valued periodic function $\sin x$. The periodicity of $\sin x$ is a manifestation of the multivalued nature of the integral defining $\sin^{-1} x$.

Similarly it turns out that the multivalued nature of elliptic integrals lead to inverse functions which are doubly periodic. After Gauss, Abel, and Jacobi, Weierstrass was the mathematician to make significant contributions in the study of elliptic integrals with his $\wp(z)$ function. In fact, similar to the example of the parametrization of a circle considered earlier, $x = \wp(z)$ and $y = \wp'(z)$ parametrize the elliptic curve $E(\mathbb{C}) : y^2 = 4x^3 - g_2 x - g_3$, with

$$z = \int_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}.$$

The function $\wp(z)$ has many remarkable properties. Suppose Λ is a lattice, i.e., a subgroup of the form $\Lambda = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$ with $\{\omega_1, \omega_2\}$ being an $\mathbb{R}$-basis for $\mathbb{C}$. A meromorphic function on $\mathbb{C}$ relative to the lattice Λ is called an elliptic function if $f(z + \omega_i) = f(z)$ for all $z \in \mathbb{C}$ and $i \in \{1, 2\}$. It turns out that $\wp(z)$ is an even elliptic function. In fact, there is another formulation of $\wp(z)$ given below.

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

with

$$g_2 = 60\,\mathrm{G}_4(\Lambda), \quad g_3 = 140\,\mathrm{G}_6(\Lambda)$$

and

$$\mathrm{G}_{2k}(\Lambda) := \sum_{\omega \in \Lambda - \{0\}} \omega^{-2k}. \tag{2.9}$$

One can define a map

$$\phi : \mathbb{C}/\Lambda \longrightarrow E(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C}), z \longleftrightarrow [\wp(z), \wp'(z), 1]$$

which is an isomorphism of Riemann surfaces that is also a group homomorphism. From the previous discussion it is clear that now we have a map from lattices to elliptic curves. Further,

$$\mathbb{C}/\Lambda \simeq \mathbb{C}/\Lambda' \iff \Lambda = c\Lambda' \quad \text{for some } c \in \mathbb{C}.$$

Also on writing $\Lambda(\tau) := \mathbb{Z}\cdot\tau + \mathbb{Z}\cdot 1 \simeq \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2$ with $\tau = \frac{\omega_1}{\omega_2}$ for some $\tau \in \mathbb{H}$, we see that $\Lambda(\tau) = \Lambda(\tau')$ if and only if there exist a matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{Z})$ such that

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

One can show that there is one-to-one correspondence between

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \leftrightarrow \text{elliptic curves over } \mathbb{C}/\simeq .$$

We now define a modular form of weight k, with respect to a congruence subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$, as a holomorphic function on the complex upper half plane $\mathbb{H}$, that satisfies the condition

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad \text{for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \tag{2.10}$$

and which is holomorphic at the cusp ∞. It can be checked that the Eisenstein series $\mathrm{G}_{2k}(\Lambda)$ defined in (2.9) is a modular form of weight $2k$.

We have given a very brief historical account of the origin of modular forms, but many important connections are already beginning to show up. In fact, since their first appearance in connection with hypergeometric equations, modular forms have become a fertile meeting ground for various subfields of mathematics such as elliptic curves, quadratic forms, quaternion algebras, Riemann surfaces, algebraic geometry, algebraic topology, to name but a few, with fruitful consequences. One such example is the celebrated modularity theorem (formerly Taniyama, Shimura, Weil conjecture), which states that every elliptic curve defined over $\mathbb{Q}$ is modular. The proof of a version of the modularity theorem by Wiles resulted in the resolution of Fermat's last theorem, the single most famous problem in number theory. The classical ideas of modular forms were further extended in a more general setting using representation theoretic tools by Gelfand, Ilya Piatetski-Shapiro and others in the 1960s resulting in the modern theory of automorphism forms. Langlands created a general theory of Eisenstein series and produced some far reaching and deep conjectures. Automorphic forms play an important role in modern number theory. In this context the following remark of Langlands on automorphic forms comes to mind- "It is a deeper subject than I appreciated and, I begin to suspect, deeper than anyone yet appreciates. To see it whole is certainly a daunting, for the moment even impossible, task.".

2.2 Klingen Eisenstein series

We have already encountered Eisenstein series as our first concrete example of modular forms. Eisenstein series encode the continuous spectrum of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ and are important objects. In fact, the origin of the Langlands functoriality conjectures can be traced back to his work on Eisenstein series and their constant terms that he carried out around 1965.

Siegel generalized the classical holomorphic Eisenstein series to higher dimensional Siegel spaces. Klingen further generalized the Siegel Eisenstein series to obtain, what is known in his honor as, the Klingen Eisenstein series [14]. Suppose f is an elliptic cusp form of weight k, then the Klingen Eisenstein series in the degree

2 case is defined as

$$E^k_{2,1}(Z,f) = \sum_{m \in C_{2,1} \backslash \mathrm{Sp}(4,\mathbb{Z})} f(m\langle Z\rangle^*) \det(j(m,Z))^{-k}. \tag{2.11}$$

Here, $Z \in \mathbb{H}_2 = \{z \in M_2(\mathbb{C}) \mid {}^t z = z, \operatorname{Im} z > 0\}$, $C_{2,1} := \{\begin{bmatrix} * & & * & * \\ * & * & * & * \\ * & & * & * \\ & & & * \end{bmatrix} \in \mathrm{Sp}(4,\mathbb{Z})\}$, for $\mathrm{Sp}(4,\mathbb{Z}) \ni m = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, $m\langle Z\rangle := (AZ+B)(CZ+D)^{-1}$, $j(m,Z) := CZ+D$, $m\langle Z\rangle^* = \tilde{\tau}$ for $m\langle Z\rangle = \begin{bmatrix} \tilde{\tau} & \tilde{z} \\ \tilde{z} & \tilde{\tau}' \end{bmatrix}$. Klingen proved that if $k \geq 6$ is an even integer then the series defined in (2.11) converges absolutely and uniformly on any vertical strip of positive height.

Chapter 3

Eisenstein series using local newforms

3.1 Introduction

In this chapter we construct a classical Eisenstein series via a global distinguished vector using local newforms. The results as well as the methods used in this chapter are well known, but perhaps some of the explicit calculations are new. In any case, this chapter serves as a good starting point for illustrating the techniques in an easier setting, which shall be used later for obtaining the main results of this work.

3.2 Eisenstein series in the adelic setting

In this section we briefly review the construction of adelic Eisenstein series for the group $\mathrm{G}(\mathbb{A}) = \mathrm{GL}(2, \mathbb{A})$. We refer readers to [23] for the general case of an arbitrary reductive group.

Let $\mathrm{B}(\mathbb{A})$ be the Borel subgroup of $\mathrm{G}(\mathbb{A})$. Let χ_1 and χ_2 be continuous characters of ideles. Let χ be the continuous character of $\mathrm{B}(\mathbb{A})$ defined as

$$\chi(\begin{bmatrix} a & b \\ & d \end{bmatrix}) = \chi_1(a)\chi_2(d) \qquad \text{for } \begin{bmatrix} a & b \\ & d \end{bmatrix} \in \mathrm{B}(\mathbb{A}).$$

Let $\mathrm{ind}_{\mathrm{B}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\chi)$ be the representation of $\mathrm{G}(\mathbb{A})$ induced form the character χ, consisting of smooth functions $f : \mathrm{G}(\mathbb{A}) \to \mathbb{C}$ which satisfy the transformation property

$$f(\begin{bmatrix} a & b \\ & d \end{bmatrix} g) = \left|\frac{a}{d}\right|^{\frac{1}{2}} \chi_1(a)\chi_2(d)f(g) \qquad \text{for all } \begin{bmatrix} a & b \\ & d \end{bmatrix} \in \mathrm{B}(\mathbb{A}), g \in \mathrm{G}(\mathbb{A}),$$

and certain other regularity conditions (like K-finiteness, see Chapter 3 in [5]). Now for f as above we define the Eisenstein series

$$E(g, f) := \sum_{\gamma \in \mathrm{B}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{Q})} f(\gamma g), \qquad \text{for } g \in \mathrm{G}(\mathbb{A}), \tag{3.1}$$

provided the sum is convergent. We may also write

$$E(g, f) = \sum_{\gamma \in \Gamma_\infty^2(\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{Z})} f(\gamma g). \tag{3.2}$$

Let π be an irreducible and admissible constituent of $\mathrm{ind}_{\mathrm{B}(\mathbb{A})}^{\mathrm{G}(\mathbb{A})}(\chi)$. One can decompose π using the tensor product theorem as $\pi \cong \bigotimes_{p \leq \infty} \pi_p$. Here π_p are representations of the local groups $\mathrm{GL}(2, \mathbb{Q}_p)$ and are irreducible and admissible constituents of the local induced representations $\mathrm{ind}_{\mathrm{B}(\mathbb{Q}_p)}^{\mathrm{G}(\mathbb{Q}_p)}(\chi_p)$. Also almost all π_p are unramified. Next we will use some special (or distinguished) vectors in local representations π_p to construct a classical Eisenstein series, showing the connection between the adelic and the classical formulations of Eisenstein series.

3.3 A calculation in degree one

Let χ_1 and χ_2 be primitive Dirichlet characters of conductors N_1 and N_2 respectively. We consider χ_1 and χ_2 as continuous characters of ideles. For $i = 1, 2$ let $\chi_i = \otimes_p \chi_{i,p}$ be the decomposition of χ_i into local components. We consider an irreducible admissible representation π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$ which is induced from the Borel subgroup via the characters $|\cdot|^{s_1}\chi_{1,p}$ and $|\cdot|^{-s_1}\chi_{2,p}$ with $s_1 \in \mathbb{C}$. Let $n_{1,p} = n_1$ and $n_{2,p} = n_2$ be the conductors of $\chi_{1,p}$ and $\chi_{2,p}$ respectively. Then $n = n_1 + n_2$ is the conductor of a local newform, i.e., n is the smallest integer such that there exists a non-trivial vector that is invariant under

$$\mathrm{K}_\mathrm{p}^2(n) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}_p)\colon c \in p^n\mathbb{Z}_p,\, d \in 1 + p^n\mathbb{Z}_p \right\}.$$

Following (20) in [35], we write a newform explicitly as,

$$f_p(g) = \begin{cases} \left|\frac{a}{d}\right|_p^{s_1+\frac{1}{2}} \chi_{1,p}(p^{-n_2})\chi_{1,p}(a)\chi_{2,p}(d) & \text{if } g \in \left[\begin{smallmatrix} a & * \\ & d \end{smallmatrix}\right]\gamma_{n_2}\,\mathrm{K}_\mathrm{p}^2(n) \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

where $\gamma_{n_2} = \begin{bmatrix} 1 & \\ p^{n_2} & 1 \end{bmatrix}$. Now, for the prime p we pick a local distinguished vector ϕ_p such that it is invariant under

$$\Gamma_0(n_1, n_2) := \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}_p)\colon a \in \mathbb{Z}_p^\times,$$
$$b \in p^{n_2}\mathbb{Z}_p,\, c \in p^{n_1}\mathbb{Z}_p,\, d \in 1 + p^{(n_1+n_2)}\mathbb{Z}_p\}.$$

Here we note that the above definition is a local analog of the group $\Gamma_0(n_1, n_2)$ defined in (7.1.3) of [20]. A simple calculation shows that the local distinguished vector is given by

$$\phi_p(g) := \pi_p\left(\begin{bmatrix} p^{n_2} & \\ & 1\end{bmatrix}\right) f_p(g) = f_p\left(g\begin{bmatrix} p^{n_2} & \\ & 1\end{bmatrix}\right). \tag{3.4}$$

Lemma 3.3.1. *Let the newform vector f_p be defined as above. Then for* $\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ *with $c \neq 0$, $n_2 > 0$ and $n_1 > 0$ we have*

$$f_p\left(\begin{bmatrix} p^{-n_2} & \\ & 1\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} p^{n_2} & \\ & 1\end{bmatrix}\right) = \begin{cases} \chi_{1,p}(p^{-n_2}c^{-1})\,\chi_{2,p}(d) & \text{if } p \nmid c,\ p \nmid d \\ 0 & \text{if } p|c \text{ or } p|d \end{cases} \tag{3.5}$$

Proof. First, we consider the case $p \nmid c$ and $p \nmid d$. Then we have

$$\begin{aligned} f_p\left(\begin{bmatrix} p^{-n_2} & \\ & 1\end{bmatrix}\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} p^{n_2} & \\ & 1\end{bmatrix}\right) &= f_p\left(\begin{bmatrix} a & p^{-n_2}b \\ p^{n_2}c & d\end{bmatrix}\right) \\ &= f_p\left(\begin{bmatrix} c^{-1} & p^{-n_2}b \\ & d\end{bmatrix}\begin{bmatrix} 1 & \\ p^{n_2} & 1\end{bmatrix}\begin{bmatrix} d^{-1}c & \\ & 1\end{bmatrix}\right) \\ &= \left|\frac{c^{-1}}{d}\right|_p^{s_1+\frac{1}{2}} \chi_{1,p}(p^{-n_2})\,\chi_{1,p}(c^{-1})\,\chi_{2,p}(d) \\ &= \chi_{1,p}(p^{-n_2})\,\chi_{1,p}(c^{-1})\,\chi_{2,p}(d). \end{aligned}$$

Next, we consider the case $p|c$ and $p \nmid d$. Let the valuation of c be m. Then,

$$f_p\left(\begin{bmatrix} a & p^{-n_2}b \\ p^{n_2}c & d\end{bmatrix}\right) = f_p\left(\begin{bmatrix} p^m c^{-1} & p^{-n_2}b \\ & d\end{bmatrix}\begin{bmatrix} 1 & \\ p^{n_2+m} & 1\end{bmatrix}\begin{bmatrix} p^{-m}cd^{-1} & \\ & 1\end{bmatrix}\right) = 0.$$

We note that the hypothesis $n_1 > 0$ is needed for the above equality to hold, otherwise if $n_1 = 0$ then we get a different result (see (3.7) below).

Next, we consider the case $p \nmid c$ and $p|d$. In this case, suppose the valuation of d is

$v(d) = m$ with $0 < m \le n_2$. Then we obtain,

$$f_p\left(\begin{bmatrix} a & p^{-n_2}b \\ p^{n_2}c & d \end{bmatrix}\right) = f_p\left(\begin{bmatrix} 1 & \\ & p^m \end{bmatrix}\begin{bmatrix} a & p^{-n_2}b \\ p^{n_2-m}c & p^{-m}d \end{bmatrix}\right)$$
$$= p^{m(s_1+\frac{1}{2})}\chi_{2,p}(p^m) f_p\left(\begin{bmatrix} p^{-m}c^{-1} & p^{-n_2}b \\ & p^{-m}d \end{bmatrix}\begin{bmatrix} 1 & \\ p^{n_2-m} & 1 \end{bmatrix}\begin{bmatrix} p^m cd^{-1} & \\ & 1 \end{bmatrix}\right)$$
$$= 0.$$

Finally, we are still assuming $p \nmid c$ and $p|d$. Let $m =$ the valuation of d. Further, suppose $m > n_2$ or $d = 0$. Then

$$f_p\left(\begin{bmatrix} a & p^{-n_2}b \\ p^{n_2}c & d \end{bmatrix}\right) = f_p\left(\begin{bmatrix} 1 & \\ & p^{n_2} \end{bmatrix}\begin{bmatrix} a & p^{-n_2}b \\ c & dp^{-n_2} \end{bmatrix}\right) = p^{n_2(s_1+\frac{1}{2})}\chi_{2,p}(p^{n_2})$$
$$f_p\left(\begin{bmatrix} c^{-1}p^{-n_2} & a - p^{-n_2}c^{-1}(1+p^n) \\ & c \end{bmatrix}\begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1+p^n & (1+p^n)c^{-1}dp^{-n_2} - 1 \\ -p^n & 1 - p^n c^{-1}dp^{-n_2} \end{bmatrix}\right)$$

(where $n = n_1 + n_2$)

$$= 0.$$

This completes the proof. $\square$

The proposition proved below characterizes the distinguished vector ϕ_p depending on the conductors of local characters.

Proposition 3.3.2. *Let the conjugated newform vector ϕ_p be defined as in (3.4) and let $n_{1,p} = n_1$ and $n_{2,p} = n_2$ be conductors of $\chi_{1,p}$ and $\chi_{2,p}$ respectively. Then for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ with $c \neq 0$,*

i. if $n_2 > 0$ and $n_1 > 0$ then

$$\phi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{cases} p^{-n_2(s_1+\frac{1}{2})}\chi_{1,p}(c^{-1})\chi_{2,p}(d) & \text{if } p \nmid c \text{ and } p \nmid d, \\ 0 & \text{if } p|c \text{ or } p|d. \end{cases} \tag{3.6}$$

ii. *if* $n_2 > 0$ *and* $n_1 = 0$

$$\phi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{cases} p^{-n_2(s_1+\frac{1}{2})}\chi_{2,p}(d) & \text{if } p \nmid d, \\ \\ 0 & \text{if } p|d. \end{cases} \tag{3.7}$$

iii. *if* $n_2 = 0$ *and* $n_1 > 0$

$$\phi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{cases} \chi_{1,p}(c^{-1}) & \text{if } p \nmid c, \\ \\ 0 & \text{if } p|c. \end{cases} \tag{3.8}$$

iv. *if* $n_2 = 0$ *and* $n_1 = 0$

$$\phi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 1. \tag{3.9}$$

Proof. For the first part of the proposition we have,

$$\begin{aligned}
\phi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) &= f_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p^{n_2} & \\ & 1 \end{bmatrix}) \\
&= f_p(\begin{bmatrix} p^{n_2} & \\ & 1 \end{bmatrix} \begin{bmatrix} p^{-n_2} & \\ & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p^{n_2} & \\ & 1 \end{bmatrix}) \\
&= |p^{n_2}|_p^{(s_1+\frac{1}{2})} \chi_{1,p}(p^{n_2}) f_p(\begin{bmatrix} p^{-n_2} & \\ & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p^{n_2} & \\ & 1 \end{bmatrix}) \\
&= p^{-n_2(s_1+\frac{1}{2})} \chi_{1,p}(p^{n_2}) \chi_{1,p}(p^{-n_2})\, \chi_{1,p}(c^{-1})\, \chi_{2,p}(d) \\
&= p^{-n_2(s_1+\frac{1}{2})}\, \chi_{1,p}(c^{-1})\, \chi_{2,p}(d).
\end{aligned}$$

We note that the second last equality follows from the Lemma 3.3.1.

Next, for the first case of the second part suppose $p \nmid d$. Then $\begin{bmatrix} & 1 \\ cd^{-1}p^{n_2} & 1 \end{bmatrix} \in$ $\mathrm{K}_{\mathrm{p}}^2(n)$, so we obtain,

$$\phi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = f_p(\begin{bmatrix} ap^{n_2} & b \\ cp^{n_2} & d \end{bmatrix})$$

$$
\begin{aligned}
&= f_p(\begin{bmatrix} d^{-1}p^{n_2} & b \\ & d \end{bmatrix} \begin{bmatrix} 1 & \\ cd^{-1}p^{n_2} & 1 \end{bmatrix}) \\
&= f_p(\begin{bmatrix} d^{-1}p^{n_2} & b \\ & d \end{bmatrix}) \\
&= p^{-n_2(s_1+\frac{1}{2})}\chi_{2,p}(d).
\end{aligned}
$$

For convenience, let $w = cd^{-1}p^{n_2}$ and let the valuation of w be i. Next, to conclude the proof of the second part we need to show that if $p \mid d$ then

$$\phi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = 0.$$

So, suppose $p \mid d$. Then $v(w) = i < n_2$. Now the desired result follows from the following observation.

$$
\begin{aligned}
f_p(\begin{bmatrix} 1 & \\ w & 1 \end{bmatrix}) &= f_p(\begin{bmatrix} p^i w^{-1} & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ p^i & 1 \end{bmatrix} \begin{bmatrix} p^{-i}w & \\ & 1 \end{bmatrix}) \\
&= 0.
\end{aligned}
$$

This completes the proof of the second part.

For the third part we note that, if $n_2 = 0$ and $n_1 > 0$ then the distinguished vector ϕ_p is same as the newform f_p and then the result follows from the Lemma 2.1.1 and Proposition 2.1.2 of [35].

The fourth part follows from the fact that under the given hypotheses f_p is $\mathrm{K}_\mathrm{p}^2(0)$ invariant. □

Now, for $p = \infty$ we pick ϕ_∞ as a weight k vector defined as

$$\phi_\infty(\begin{bmatrix} a & b \\ & d \end{bmatrix} r(\theta)) = \left|\frac{a}{d}\right|_\infty^{s_1+\frac{1}{2}} \exp(ik\theta) \qquad \text{for all } a, d \in \mathbb{R}^\times, b, \theta \in \mathbb{R}. \tag{3.10}$$

Next, we set $s_1 = s + \frac{k-1}{2}$. We also assume that

$$\chi_1^{-1}(-1)\chi_2(-1) = (-1)^k. \tag{3.11}$$

Next, we define our global distinguished vector

$$\phi := \phi_\infty \otimes \bigotimes_{p<\infty} \phi_p\,. \tag{3.12}$$

Let $g = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix}$, with $x, y \in \mathbb{R}$, $y > 0$. Before proceeding further, we note a simple matrix identity which will be useful later.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} y^{1/2} & \\ & y^{-1/2} \end{bmatrix} = \begin{bmatrix} 1 & x' \\ & 1 \end{bmatrix} \begin{bmatrix} \sqrt{y'} & \\ & \sqrt{y'}^{-1} \end{bmatrix} r(\theta) \tag{3.13}$$

where,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}),\ x' + iy' = \frac{a\tau + b}{c\tau + d} \text{ and } \exp(i\theta) = \frac{c\bar{\tau} + d}{|c\tau + d|}.$$

Then we obtain an Eisenstein series, using the global distinguished vector ϕ defined in (3.12) and with a convenient normalizing factor of $y^{-(s+\frac{k}{2})}$, as follows.

$$\begin{aligned}
E(\phi, g) &= y^{-(s+\frac{k}{2})} \sum_{\gamma \in \Gamma^2_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \phi(\gamma g) \\
&= y^{-(s+\frac{k}{2})} \sum_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma^2_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \left(\phi_\infty \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} g \right) \prod_{p<\infty} \phi_p \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right) \\
&= y^{-(s+\frac{k}{2})} \sum_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma^2_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} {N_2}^{-(s+\frac{k}{2})} \chi_1^{-1}(c)\chi_2(d)\, \phi_\infty \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} g \right)
\end{aligned}$$

(The above equality follows precisely from Proposition 3.3.2.)

$$\stackrel{(3.13)}{=} y^{-(s+\frac{k}{2})} \sum_{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma^2_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} {N_2}^{-(s+\frac{k}{2})} \chi_1^{-1}(c)\chi_2(d)$$

$$
\begin{aligned}
&\phi_\infty\left(\begin{bmatrix}1 & x'\\ & 1\end{bmatrix}\begin{bmatrix}\sqrt{y'} & \\ & \sqrt{y'}^{-1}\end{bmatrix} r(\theta)\right)\\
&= y^{-(s+\frac{k}{2})} \sum_{\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]\in\Gamma^2_\infty\backslash \mathrm{SL}(2,\mathbb{Z})} {N_2}^{-(s+\frac{k}{2})}\, \chi_1^{-1}(c)\chi_2(d)\\
&\qquad \left(\frac{y}{|c\tau+d|^2}\right)^{(s+\frac{k}{2})} \left(\frac{c\tau+d}{|c\tau+d|}\right)^{-k}\\
&= \sum_{\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right]\in\Gamma^2_\infty\backslash \mathrm{SL}(2,\mathbb{Z})} {N_2}^{-(s+\frac{k}{2})}\, \chi_1^{-1}(c)\chi_2(d)\, |c\tau+d|^{-2s}\, (c\tau+d)^{-k}\\
&= \sum_{\{\pm1\}\backslash\{(c,d)\in\mathbb{Z}\times\mathbb{Z}\,|\,\gcd(c,d)=1\}} {N_2}^{-(s+\frac{k}{2})}\, \chi_1^{-1}(c)\chi_2(d)\, |c\tau+d|^{-2s}\, (c\tau+d)^{-k}\\
&=^{(3.11)} \frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\ \gcd(c,d)=1}} {N_2}^{-(s+\frac{k}{2})}\, \chi_1^{-1}(c)\chi_2(d)\, |c\tau+d|^{-2s}\, (c\tau+d)^{-k}\\
&= \frac{1}{2{N_2}^{(s+\frac{k}{2})}\ \mathrm{L}(\chi_1^{-1}\chi_2, 2s+k)} \sum_{\substack{(c,d)\in\mathbb{Z}\times\mathbb{Z}\\ (c,d)\neq(0,0)}} \chi_1^{-1}(c)\chi_2(d)\, |c\tau+d|^{-2s}\, (c\tau+d)^{-k},
\end{aligned}
$$

where as usual

$$
\mathrm{L}(\chi_1^{-1}\chi_2, 2s+k) = \sum_{n=1}^{\infty} \frac{\chi_1^{-1}\chi_2(n)}{n^{2s+k}}.
$$

We see that after taking into account the Dirichlet L-function factor noted above, in the region of convergence $k + 2\,\mathrm{Re}\, s > 2$, the Eisenstein series defined in (7.2.1) of [20] corresponds to the distinguished vector $\phi = \otimes\phi_p$ in the global induced representation.

Chapter 4

Klingen Eisenstein series

In this chapter we show the existence of a vector, in a parabolically induced global representation, that corresponds to a given classical Klingen Eisenstein series.

4.1 Global induction

We describe two different models of the global induced representations of $\mathrm{GSp}(4,\mathbb{A})$ and we also show that a $\mathbb{C}$-valued function Φ could be realized as a vector in one of the models. In this discussion we closely follow Section 2.3 of [25]. Let (π, V_π) be a cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ and let χ be a character of ideles. Let us denote by $\chi \rtimes \pi$ the induced representation of $\mathrm{GSp}(4,\mathbb{A})$ that is obtained by $Q(\mathbb{A})$ via normalized parabolic induction. The space of $\chi \rtimes \pi$ consists of functions $\tilde{\phi}\colon \mathrm{GSp}(4,\mathbb{A}) \to V_\pi$ with the transformation property

$$\tilde{\phi}(hg) = |t^2\,(ad-bc)^{-1}|\,\chi(t)\,\pi(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\,\tilde{\phi}(g),$$

$$\text{for } h = \begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} \in Q(\mathbb{A}). \tag{4.1}$$

Let us denote by $I(s,\pi)$ the representation $|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$. To each $\tilde{\phi} \in I(s,\pi)$ we may associate the $\mathbb{C}$-valued function Φ such that $\Phi(g) = \tilde{\phi}(g)(1)$. Let $I_{\mathbb{C}}(s,\pi)$ be the model of $I(s,\pi)$ thus obtained.

Let $Z \in \mathbb{H}_2$ such that

$$Z = \begin{bmatrix} \tau & z \\ z & \tau' \end{bmatrix}, \qquad \tau = x + iy, \qquad z = u + iv, \qquad \tau' = x' + iy', \tag{4.2}$$

where x, y, u, v, x', y' are real numbers, $y, y' > 0$, and $yy' - v^2 > 0$. Let

$$b_Z = \begin{bmatrix} 1 & & x & u \\ & 1 & u & x' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ v/y & 1 & & \\ & & 1 & -v/y \\ & & & 1 \end{bmatrix} \begin{bmatrix} b & & & \\ & a & & \\ & & b^{-1} & \\ & & & a^{-1} \end{bmatrix} \tag{4.3}$$

with

$$a = \sqrt{y' - \frac{v^2}{y}} \quad \text{and} \quad b = \sqrt{y}. \tag{4.4}$$

It can be checked that

$$b_Z\langle I\rangle = Z. \tag{4.5}$$

Theorem 4.1.1. *Let k be an even integer. Let f be an elliptic cusp form of weight k. Let ϕ be the automorphic form associated with f and let (π, V_π) be the irreducible cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A})$ generated by ϕ. Then there exists a function $\Phi : \mathrm{GSp}(4,\mathbb{A}) \to \mathbb{C}$ in the model $I_{\mathbb{C}}(s,\pi)$ of the global induced representation $|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$ of $\mathrm{GSp}(4,\mathbb{A})$ with $s = k-2$ such that*

$$F(Z,s) := \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \Phi(\gamma b_Z), \tag{4.6}$$

$$(\textit{with } b_Z \in B(\mathbb{R}) \textit{ as defined in } (4.3), \textit{ so that } b_Z\langle\left[\begin{smallmatrix} i & \\ & i \end{smallmatrix}\right]\rangle = Z),$$

defines an Eisenstein series that is exactly the same as the classical Klingen Eisen-

stein series defined in (2.11) in its region of convergence, i.e., for $k \geq 6$.

Proof. It is given that $\phi \in (\pi, V_\pi)$ is the cuspidal automorphic form associated with f. It means ϕ is a function defined on $\mathrm{GL}(2,\mathbb{A})$ such that for $g = \gamma g_\infty k_g$ with $\gamma \in \mathrm{GL}(2,\mathbb{Q})$, $g_\infty \in \mathrm{GL}(2,\mathbb{R}^+)$ and $k_g \in \prod_{p<\infty} \mathrm{GL}(2,\mathbb{Z}_p)$

$$\phi(g) = (f|_k g_\infty)(i). \tag{4.7}$$

Here we note that it is a consequence of the strong approximation theorem that any $g \in \mathrm{GL}(2,\mathbb{A})$ can be written as $g = \gamma g_\infty k_g$ for some $\gamma \in \mathrm{GL}(2,\mathbb{Q})$, $g_\infty \in \mathrm{GL}(2,\mathbb{R}^+)$ and $k_g \in \prod_{p<\infty} \mathrm{GL}(2,\mathbb{Z}_p)$. From (4.7) it follows that for $x + iy \in \mathbb{H}$

$$y^{-k/2}\phi\left(\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{y} & \\ & (\sqrt{y})^{-1}\end{bmatrix}\right) = f(x+iy) \tag{4.8}$$

Moreover, ϕ has the following properties.

1. $\phi(\gamma g) = \phi(g) \quad$ for all $\gamma \in \mathrm{GL}(2,\mathbb{Q})$, $g \in \mathrm{GL}(2,\mathbb{A})$,
2. $\phi(gr(\theta)) = \exp(ik\theta)\phi(g)$, where $r(\theta) = \left[\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right]$,
3. $\phi(gk_p) = \phi(g)$, for $k_p \in \mathrm{GL}(2,\mathbb{Z}_p)$,
4. ϕ is smooth, of moderate growth, and it satisfies other 'nice' properties of an automorphic form.
5. ϕ is cuspidal, i.e., $\int_{\mathbb{Q}\backslash\mathbb{A}} \phi(\left[\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}\right] g)\, dx = 0$ for every $g \in \mathrm{GL}(2,\mathbb{A})$.

Next we want to lift ϕ to define an automorphic form on $\mathrm{GSp}(4,\mathbb{A})$. We define $\tilde{\phi} : \mathrm{GSp}(4,\mathbb{A}) \to V_\pi$ as

$$\tilde{\phi}(g)(\begin{bmatrix}a & b\\ c & d\end{bmatrix}) = |t_1|^k\, |a_1d_1 - b_1c_1|^{-\frac{k}{2}} \det(j(k_{1\infty}, I))^{-k}\phi(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}a_1 & b_1\\ c_1 & d_1\end{bmatrix}), \tag{4.9}$$

where $g = h_1 k_1$ with

$$h_1 = \begin{bmatrix} a_1 & & b_1 & * \\ * & t_1 & * & * \\ c_1 & & d_1 & * \\ & & & t_1^{-1}(a_1d_1 - b_1c_1) \end{bmatrix} \in Q(\mathbb{A}), \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{A}) \tag{4.10}$$

and k_1 is in the standard maximal compact subgroup of $\mathrm{GSp}(4, \mathbb{A})$, $k_{1\infty}$ is the archimedean component of k_1 and $I := [{}^{i}{}_{i}]$. Here we have used the Iwasawa decomposition to write $g = h_1 k_1$. We need to check that $\tilde{\phi}$ is well-defined. Suppose $q_1 k_1 = q_2 k_2$ where $q_1, q_2 \in Q(\mathbb{A})$ and $k_1, k_2 \in \kappa$, with κ denoting the standard maximal compact subgroup of $\mathrm{GSp}(4, \mathbb{A})$. We need to show that $\tilde{\phi}(q_1k_1) = \tilde{\phi}(q_2k_2)$. Let

$$q_i = \begin{bmatrix} a_i & & b_i & * \\ * & t_i & * & * \\ c_i & & d_i & * \\ & & & t_i^{-1}(a_id_i - b_ic_i) \end{bmatrix} \quad \text{for } i = 1, 2 \text{ and}$$

$$k_2k_1^{-1} = \begin{bmatrix} \tilde{a} & & \tilde{b} & * \\ * & \tilde{t} & * & * \\ \tilde{c} & & \tilde{d} & * \\ & & & (\tilde{t})^{-1}(\tilde{a}\tilde{d} - \tilde{b}\tilde{c}) \end{bmatrix}.$$

We note that $k_2k_1^{-1} \in \kappa$ means that $(k_2k_1^{-1})_\infty$ consists of the matrices of the form $\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in \mathrm{Sp}(4, \mathbb{R})$. The symplectic conditions then imply that for some $\theta \in [0, 2\pi]$ the matrix $\begin{bmatrix} \tilde{a}_\infty & \tilde{b}_\infty \\ \tilde{c}_\infty & \tilde{d}_\infty \end{bmatrix}$ is of the form $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. This means that $|\tilde{a}\tilde{d} - \tilde{b}\tilde{c}| = 1$, because we also have that at any non-archimedean place p, $\begin{bmatrix} \tilde{a}_p & \tilde{b}_p \\ \tilde{c}_p & \tilde{d}_p \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}_p)$. We now calculate for $h \in \mathrm{GL}(2, \mathbb{A})$

$$\begin{aligned} \tilde{\phi}(q_1k_1)(h) &= \phi\left(h \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right))|t_1|^k |a_1d_1 - b_1c_1|^{-k/2} \det(j(k_{1\infty}, I))^{-k} \\ &= \phi\left(h \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix}\right))|t_2\tilde{t}|^k |a_2d_2 - b_2c_2|^{-k/2} |\tilde{a}\tilde{d} - \tilde{b}\tilde{c}|^{-k/2} \\ &\qquad \det(j(k_{1\infty}, I))^{-k} \end{aligned}$$

$$= \phi\left(h \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \tilde{a}_\infty & \tilde{b}_\infty \\ \tilde{c}_\infty & \tilde{d}_\infty \end{bmatrix}\right))|t_2\tilde{t}|^k |a_2d_2 - b_2c_2|^{-k/2} \det(j(k_{1\infty}, I))^{-k}$$

$$= \exp(ik\theta)\phi\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) |t_2\tilde{t}|^k |a_2d_2 - b_2c_2|^{-k/2}$$

$$\big(\exp(i\theta)|\tilde{t}|\det(j(k_{2\infty}, I))\big)^{-k}$$

$$= \phi\left(h \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right))|t_2|^k |a_2d_2 - b_2c_2|^{-k/2} \det(j(k_{2\infty}, I))^{-k}$$

$$= \tilde{\phi}(q_2k_2)(h).$$

This shows that $\tilde{\phi}$ is well-defined. It is also easy to see that $\tilde{\phi} \in I(k-2, \pi)$. Now associated to $\tilde{\phi}$ we define the function $\Phi : \mathrm{GSp}(4, \mathbb{A}) \to \mathbb{C}$ given by $\Phi(g) = \tilde{\phi}(g)(1)$. We write Φ more explicitly by unraveling its definition as

$$\Phi(h_1k_1) = \Phi\left(\begin{bmatrix} a_1 & & b_1 & * \\ * & t_1 & * & * \\ c_1 & & d_1 & * \\ & & & t_1^{-1}(a_1d_1 - b_1c_1) \end{bmatrix} k_1\right) = \phi\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) |a_1d_1 - b_1c_1|^{-k/2}$$

$$|t_1|^k \det(j(k_{1\infty}, I))^{-k}, \tag{4.11}$$

with h_1 as in (4.10) and $k_1 \in \kappa$.

Next we write $Z = X + iY$ and for $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$ we set $\tilde{Z} = \tilde{X} + i\tilde{Y} = \gamma\langle Z \rangle$. A straightforward calculation shows that

$$\tilde{Y} = {}^t(CZ + D)^{-1} Y (C\bar{Z} + D)^{-1}, \quad \gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{4.12}$$

For $Z = I$ and $\gamma = b_Z$ it follows that

$$\det(j(b_Z, I)) = \det(Y)^{-1/2}. \tag{4.13}$$

Let $\gamma \in \mathrm{Sp}(4, \mathbb{R})$, $Z \in \mathbb{H}_2$, $\tilde{Z} = \gamma\langle Z \rangle$ and $b_{\tilde{Z}} \in Q(\mathbb{R})$ such that $b_{\tilde{Z}}\langle I \rangle = \tilde{Z}$. Then it follows from (4.12) and (4.13) that $\gamma b_Z = b_{\tilde{Z}}\kappa_\infty$ with

$$\det(j(\kappa_\infty, I)) = \sqrt{\frac{\det \tilde{Y}}{\det Y}} \det(j(\gamma, Z)). \tag{4.14}$$

Let us define, for $g \in \mathrm{GSp}(4, \mathbb{A})$

$$E(\Phi, g) := \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \Phi(\gamma g). \tag{4.15}$$

Since from the strong approximation for $\mathrm{GSp}(4, \mathbb{A})$ and the definition of Φ as in (4.11) it follows that Φ is determined only on $\mathrm{GSp}(4, \mathbb{R})$, we define

$$F(Z, s) := \det(Y)^{-k/2} E(\Phi, b_Z) = \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \Phi(\gamma b_Z). \tag{4.16}$$

We have,

$$\begin{aligned} F(Z, s) &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \Phi(\gamma b_Z) \\ &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z})} \Phi(\gamma b_Z) \\ &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z})} \Phi(b_{\tilde{Z}}\kappa_\infty). \end{aligned}$$

Let

$$b_{\tilde{Z}} = \begin{bmatrix} 1 & & \tilde{x} & \tilde{u} \\ & 1 & \tilde{u} & \tilde{x}' \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ \tilde{v}/\tilde{y} & 1 & & \\ & & 1 & -\tilde{v}/\tilde{y} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \tilde{b} & & & \\ & \tilde{a} & & \\ & & \tilde{b}^{-1} & \\ & & & \tilde{a}^{-1} \end{bmatrix} \tag{4.17}$$

with $\tilde{a}$ and $\tilde{b}$ defined analogous to the definition of a and b in (4.4). Then,

$$F(Z, s) = \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z})} \phi\left(\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{b} & \\ & \tilde{b}^{-1} \end{bmatrix}\right) \det(j(\kappa_\infty, I))^{-k} |\tilde{a}|^k$$

$$
\begin{aligned}
&=^{(4.14)} \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z})} \phi\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & \\ & \tilde{b}^{-1}\end{bmatrix}\right)\\
&\qquad\qquad \left(\sqrt{\frac{\det \tilde{Y}}{\det Y}}\det(j(\gamma,Z))\right)^{-k} |\tilde{a}|^k\\
&=^{(4.4)} \sum_{\gamma \in Q(\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z})} \phi\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & \\ & \tilde{b}^{-1}\end{bmatrix}\right)\left(\sqrt{\det \tilde{Y}}\det(j(\gamma,Z))\right)^{-k}\\
&\qquad\qquad \left(\frac{\det(\tilde{Y})}{\tilde{y}}\right)^{k/2}\\
&= \sum_{\gamma \in Q(\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z})} \phi\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{\tilde{y}} & \\ & (\sqrt{\tilde{y}})^{-1}\end{bmatrix}\right)(\tilde{y})^{-k/2}\det(j(\gamma,Z))^{-k}\\
&= \sum_{\gamma \in Q(\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z})} f(\tilde{x}+i\tilde{y})\det(j(\gamma,Z))^{-k}\\
&= \sum_{\gamma \in Q(\mathbb{Z})\backslash \mathrm{Sp}(4,\mathbb{Z})} f(\gamma\langle Z\rangle^{*})\det(j(\gamma,Z))^{-k}
\end{aligned}
$$

Therefore we see that the Klingen Eisenstein series given by (2.11) can be obtained from the function Φ defined in (4.11). It is clear that $\Phi \in I_{\mathbb{C}}(k-2,\pi)$. □

Chapter 5

Local representations

In the previous chapter we have seen how to obtain a Klingen Eisenstein series for the full group $\mathrm{Sp}(4,\mathbb{Z})$ using the representation theoretic method of global parabolic induction. Our objective in this chapter is to understand the corresponding local representations. Assume $\mathbb{Z} \ni k \geq 4$ to be an even integer and let Π be the automorphic representation $|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$ of $\mathrm{GSp}(4,\mathbb{A})$. We know from the tensor product theorem that $\Pi \cong \bigotimes\limits_{p\leq\infty} \Pi_p$, where almost all Π_p are unramified. Similarly, $\pi \cong \bigotimes\limits_{p\leq\infty} \pi_p$. Here π_p denotes an irreducible admissible representation of $\mathrm{GL}(2,\mathbb{Q}_p)$ and almost all π_p are unramified.

5.1 The non-archimedean local representations $\nu^s \rtimes \nu^{-s/2}\pi_p$

In the notation of [30], for $p < \infty$ we are interested in the local representation $\nu^s \rtimes \nu^{-s/2}\pi_p$. The non-supercuspidal representations of $\mathrm{GSp}(4,\mathbb{Q}_p)$ are well understood and classified. We refer the reader to [30] and [32] for further details.

5.2 The archimedean story

The main goal of this section is to look at the archimedean component Π_∞ in the broader context of the representation theory of real reductive groups and to show that inside this representation there exists a distinguished vector, which shall be essential in obtaining a holomorphic Siegel modular form associated to Π. We shall describe this distinguished vector more explicitly later in this section, but first we recall some well known facts related to the representation theory of real reductive groups. Our main references for this section are [15], [16], [26] and [24]. Experts may skip most of this section and go directly to Proposition 5.2.7.

5.2.1 Preliminaries

Let G be a real reductive algebraic group. A unitary representation (π, V) of G is a norm preserving continuous group action of G on a Hilbert space V. Let (π, V) be a unitary representation of G. The set of equivalence classes of irreducible unitary representations of G is called the unitary dual of G, denoted as $\hat{G}$. An important problem in representation theory is the unitarity problem, which refers to the problem of finding $\hat{G}$ for any given G. For example, the classical theory of Fourier series can be viewed as the unitarity problem for the unitary representation of the circle group S^1 on $\mathrm{L}^2(S^1)$. In this case, as S^1 is compact, its unitary representation on $\mathrm{L}^2(S^1)$ decomposes as a discrete sum of irreducible unitary representations. In general, the analysis of $\mathrm{L}^2(G)$ consists of both a discrete part and a continuous part. Some of such early examples came from Physics. Bargmann [2] classified representations of $\mathrm{SL}(2,\mathbb{R})$ in 1947. One can already see some features of the general theory such as the appearance of the Lie algebra in his work. However, the analysis for a general real reductive group G is much more complicated. In a series of groundbreaking papers Harish-Chandra, with his deep insight, provided a framework to deal with the general case. Let K be a maximal compact subgroup of G. We say that a vector $v \in V$ is K-**finite** if $\pi(K)v$ is finite dimensional. It is a consequence of the Peter-Weyl theorem that $(\pi|_K, V_K)$ decomposes as a discrete sum

$$V_K \cong \bigoplus_{V_\lambda \in \hat{K}} n_\lambda V_\lambda \tag{5.1}$$

where $\hat{K}$ is the unitary dual of K, i.e., the set of equivalence classes of unitary irreducible representations of K and n_λ is the multiplicity with which V_λ occurs in the sum. If n_λ is positive then the corresponding equivalence class is called a K-**type** occurring in π. We say π is **admissible** if each n_λ occurring in the sum above is finite, i.e., if each K-type occurs with finite multiplicity. It turns out that irreducible unitary representations are admissible (see Theorem 8.1, [15]). Suppose $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$. Let $\mathfrak{h}^{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Then by differentiating the action of G on V, one can consider V_K as a module for the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$. Actually, V_K may be thought of as a $(\mathfrak{g}, K)$-module as clearly K acts on V_K. We say that two admissible representations of G are **infinitesimally equivalent** if the underlying $(\mathfrak{g}, K)$-modules are algebraically equivalent. Harish-Chandra proved that two irreducible unitary representations of G on Hilbert spaces are unitarily equivalent if and only if they are infinitesimally equivalent (see Theorem 0.6, [17]). This Lie algebra representation (along with K-module structure) captures complete information of the group representation, so the upshot of this approach is that in place of considering an admissible representation of G on the Hilbert space V, one could now deal with an algebraic object, namely the $(\mathfrak{g}, K)$-module V_K. Next, it is natural to consider the action of the center $\mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ of $\mathrm{U}(\mathfrak{g}^{\mathbb{C}})$. It follows from Schur's Lemma that any $z \in \mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ acts by a scalar. This scalar could be computed by considering the action of z on a non-zero highest weight vector. But first we fix some notation. Let $\mathfrak{h}$ be a real form of $\mathfrak{h}^{\mathbb{C}}$. Let Δ be a set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Let W be the Weyl group of this root system. W acts on $(\mathfrak{h}^{\mathbb{C}})' := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})$ and so it also acts on $\mathfrak{h}^{\mathbb{C}}$ via the pairing that comes from the inner product $\langle \cdot\,, \cdot \rangle$ built from $\mathfrak{g}$. This action can be further extended to an action of W on $\mathcal{H} := \mathrm{U}(\mathfrak{h}^{\mathbb{C}})$ giving an algebra automorphism of $\mathcal{H}$. Let $\mathcal{H}^W$ be the subalgebra that consists of W invariant vectors of $\mathcal{H}$.

Next we fix a system of positive roots Δ^+ and also assume that $\mathfrak{g}^{\mathbb{C}}$ has the following root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha} \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha}. \tag{5.2}$$

We define

$$\begin{aligned} \mathfrak{n}^+ &= \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{\alpha}, & \mathfrak{n}^- &= \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}, \\ \mathfrak{b} &= \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{n}^+, & \delta &= \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha. \end{aligned} \tag{5.3}$$

We know that with $\alpha \in \Delta^+$, $\dim(\mathfrak{g}_{\pm\alpha}) = 1$ and we also assume that $\mathrm{E}_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ constitute a Chevalley basis. We define

$$\mathcal{P} = \sum_{\alpha \in \Delta^+} \mathrm{U}(\mathfrak{g}^{\mathbb{C}})\, \mathrm{E}_{\alpha}, \qquad \mathcal{N} = \sum_{\alpha \in \Delta^+} \mathrm{E}_{-\alpha}\, \mathrm{U}(\mathfrak{g}^{\mathbb{C}}). \tag{5.4}$$

By Poincaré-Birkhoff-Witt theorem we can write

$$\mathrm{U}(\mathfrak{g}^{\mathbb{C}}) = \mathcal{H} \oplus \mathcal{P} \oplus \mathcal{N}. \tag{5.5}$$

Let γ_{Δ^+} be the projection of $\mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ on $\mathcal{H}$-term in (5.5). A computation shows that in order to understand the action of any $z \in \mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ on a highest weight vector, it is enough to consider the action of $\gamma_{\Delta^+}(z)$ (see Lecture 5, [16]). Harish-Chandra observed that a slight adjustment in γ_{Δ^+} leads to better symmetry properties. We define, a linear map $\sigma_{\Delta^+} : \mathfrak{h}^{\mathbb{C}} \to \mathcal{H}$ given by

$$\sigma_{\Delta^+}(H) = H - \delta(H)1, \tag{5.6}$$

and extend it to an algebra automorphism of $\mathcal{H}$ by the universal property of $\mathcal{H}$. Then we define the **Harish-Chandra homomorphism** $\gamma = \sigma_{\Delta^+} \circ \gamma_{\Delta^+}$ as a map of $\mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ into $\mathcal{H}$.

Theorem 5.2.1. *(Harish-Chandra, Theorem 8.1, [15]) The Harish-Chandra homomorphism γ is an algebra isomorphism of $\mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ onto $\mathcal{H}^W$ and it does not depend on the choice of the positive system Δ^+.*

Fix some $\lambda \in (\mathfrak{h}^{\mathbb{C}})'$. One can define an algebra homomorphism $\chi_\lambda = \lambda \circ \gamma$ from $\mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ to $\mathbb{C}$. In fact, every algebra homomorphism from $\mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ to $\mathbb{C}$ is of the form χ_λ for some $\lambda \in (\mathfrak{h}^{\mathbb{C}})'$ (see Proposition 8.21, [15]). Suppose Φ is an irreducible admissible representation of G such that $\mathrm{Z}(\mathfrak{g}^{\mathbb{C}})$ acts on the space of K-finite vectors V_K via the character $\chi = \chi_\lambda$ for some $\lambda \in (\mathfrak{h}^{\mathbb{C}})'$, then we say that the representation Φ has an **infinitesimal character** χ_λ (or sometimes λ, depending on the context). An infinitesimal character is defined up to the action of the Weyl group W, i.e., if $\lambda, \lambda' \in (\mathfrak{h}^{\mathbb{C}})'$, then $\chi_\lambda = \chi_{\lambda'}$ if and only if $\lambda = w\lambda'$ for some $w \in W$ (see Proposition 8.20, [15]).

Next we recall definitions of discrete series and tempered representations. It is well known that an irreducible unitary representation π of a unimodular group, the following three conditions are equivalent (see Proposition 9.6, [15]):

1. Some nonzero matrix coefficient is in $\mathrm{L}^2(G)$.

2. All matrix coefficient is in $\mathrm{L}^2(G)$.

3. π is equivalent to a direct summand of the right regular representation of G on $\mathrm{L}^2(G)$.

If a representation π satisfies these equivalent conditions, then we say that π is in the **discrete series** of G. If for an irreducible representation π of G all K-finite matrix coefficient are in $\mathrm{L}^{2+\epsilon}(G)$ for every $\epsilon > 0$, then we say that π is an irreducible **tempered** representation of G.

The next step towards our initial objective of classification of unitary duals of G involves a celebrated theorem of Langlands, namely the Langlands classification theorem. Roughly speaking, the Langlands classification theorem reduces the classification of unitary duals of G to the problem of classification of irreducible tempered representations of M where $P = MAN$ is the Langlands decomposition of a parabolic subgroup P of G. Further the classification of irreducible tempered representations of M depends on the data coming from certain discrete series, or limits of discrete series, representations. We provide more details in the following. We shall continue to assume G to be a linear connected reductive group. We also assume that the center of G is compact. Let $\mathfrak{a}$ be the Lie algebra of A. Suppose (σ, V) is an irreducible representation of M and ν is a member of $i\mathfrak{a}'$ then we shall denote by $\mathrm{Ind}_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$ the representation of G obtained via normalized induction (c.f. Chapter VII, [15]). Now we recall the Langlands classification theorem.

Theorem 5.2.2. *(Langlands, Theorem 8.54, [15]) Fix a minimal parabolic subgroup $S_{\mathfrak{p}} = M_{\mathfrak{p}} A_{\mathfrak{p}} N_{\mathfrak{p}}$ of G. Then the equivalence classes (under infinitesimal equivalence) of irreducible admissible representation of G stand in one-to-one corresponds with all triples $(S, [\sigma], \nu)$ such that*
$S = MAN$ is a parabolic subgroup containing $S_{\mathfrak{p}}$,
σ is an irreducible tempered representation of M and $[\sigma]$ is its equivalence class,
ν is a member of $(a')^{\mathbb{C}}$ with $\mathrm{Re}\,\nu$ in the open positive Weyl chamber.

*The correspondence is that $(MAN, [\sigma], \nu)$ corresponds to the class of the unique irreducible quotient of $Ind_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$. The parameters $(MAN, [\sigma], \nu)$ are called the **Langlands parameters** of the given representation.*

The next theorem, given below, relates the classification of irreducible tempered representations to discrete series representations.

Theorem 5.2.3. *(Theorem 8.53,[15]): For an irreducible admissible representation of G the following are equivalent:*

1. *All K-finite matrix coefficients are in $\mathrm{L}^{2+\epsilon}(G)$ for every $\epsilon > 0$.*
2. *π is infinitesimally equivalent with a subrepresentation of a standard induced representation $\mathrm{U}(S, \omega, \nu) := Ind_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$ for some parabolic subgroup $S = MAN$, a discrete series representation ω of M and an imaginary parameter ν on $\mathfrak{a}$.*

Next we note the following theorem of Harish-Chandra on parametrizations of discrete series, but first we fix some notations. Let G be a linear connected semisimple group and let K be a maximal compact subgroup of G with rank G = rank K. The equal-rank condition is essential for the existence of a discrete series. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Let $\mathfrak{b} \subset \mathfrak{k}$ be a Cartan subalgebra. Let Δ = roots of $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ and Δ_K = roots of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{b}^{\mathbb{C}})$ = the set of compact roots. Assume W_G and W_K to be the Weyl groups of Δ and Δ_K, respectively.

Theorem 5.2.4. *(Harish-Chandra, Theorem 9.20, [15]) Let G be linear connected semisimple with rank G = rank K. Suppose that $\lambda \in (i\mathfrak{b})'$ is nonsingular relative to Δ (i.e., $\langle \lambda, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$) and that Δ^+ is defined as*

$$\Delta^+ = \{\alpha \in \Delta \,|\, \langle \lambda, \alpha \rangle > 0\}. \tag{5.7}$$

Let $\delta_G = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ *and* $\delta_K = \frac{1}{2}\sum_{\alpha\in(\Delta^+\cap\Delta_K)}\alpha$*. If* $\lambda+\delta_G$ *is analytically integral, then there exits a discrete series representation* π_λ *of* G *with the following properties:*

(a) π_λ *has infinitesimally character* χ_λ*.*

(b) $\pi_\lambda|_K$ *contains with multiplicity one the* K*-type with the highest weight*

$$\Lambda = \lambda + \delta_G - 2\delta_K.$$

(c) If Λ' *is the highest weight of a* K*-type in* $\pi_\lambda|_K$*, then* Λ' *is of the form*

$$\Lambda' = \lambda + \sum_{\alpha\in\Delta^+} n_\alpha\alpha \qquad \text{for integers } n_\alpha \geq 0.$$

Two such constructed representations π_λ *are equivalent if and if only if their parameters* λ *are conjugate under* W_K*.*

λ is known as the **Harish-Chandra parameter** of the discrete series π_λ and the K-type parameter Λ is called the **Blattner parameter**. The representations described above exhaust discrete series representations, i.e., only discrete series representations of G, up to equivalence, are π_λ described in Theorem 5.2.4 (see, Theorem 12.21 [15]).

5.2.2 Discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$

After having described the well known abstract theory, we now turn our attention to the special case of our interest, which is the group $\mathrm{GSp}(4,\mathbb{R})$. In the following we shall closely follow the exposition in [25]. We shall first consider the representations of $\mathrm{Sp}(4,\mathbb{R})$ and then see how to extend these to representation of $\mathrm{GSp}(4,\mathbb{R})$.

Let us fix some notations. Let K be the standard maximal compact subgroup of

$\mathrm{Sp}(4,\mathbb{R})$. K is, in fact, isomorphic to $U(2)$ via the mapping $\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right]$ to $A+iB$. We will use the following coordinates on K (c.f., (1), [26]).

$$\mathbb{R}^4 \ni (\varphi_1, \varphi_2, \varphi_3, \varphi_4) \longmapsto r_1(\varphi_1) r_2(\varphi_2) r_3(\varphi_3) r_4(\varphi_4) \in K, \tag{5.8}$$

where

$$r_1(\varphi_1) = \begin{bmatrix} \cos(\varphi_1) & \sin(\varphi_1) & & \\ -\sin(\varphi_1) & \cos(\varphi_1) & & \\ & & \cos(\varphi_1) & \sin(\varphi_1) \\ & & -\sin(\varphi_1) & \cos(\varphi_1) \end{bmatrix},$$
$$r_2(\varphi_2) = \begin{bmatrix} \cos(\varphi_2) & & & \sin(\varphi_2) \\ & \cos(\varphi_2) & \sin(\varphi_2) & \\ & -\sin(\varphi_2) & \cos(\varphi_2) & \\ -\sin(\varphi_2) & & & \cos(\varphi_2) \end{bmatrix},$$
$$r_3(\varphi_3) = \begin{bmatrix} \cos(\varphi_3) & & \sin(\varphi_3) & \\ & 1 & & \\ -\sin(\varphi_3) & & \cos(\varphi_3) & \\ & & & 1 \end{bmatrix},$$
$$r_4(\varphi_4) = \begin{bmatrix} 1 & & & \\ & \cos(\varphi_4) & & \sin(\varphi_4) \\ & & 1 & \\ & -\sin(\varphi_4) & & \cos(\varphi_4) \end{bmatrix}.$$

Definition 5.2.5. *Let p, t be integers, and if Ψ is a function on K with the property that $k = r_3(\varphi_3) r_4(\varphi_4) \in K$ acts on Ψ as follows*

$$k.\Psi(g) = \Psi(g r_3(\varphi_3) r_4(\varphi_4)) = e^{i(p\varphi_3 + t\varphi_4)} \Psi(g) \qquad \textit{for all } \varphi_3, \varphi_4 \in \mathbb{R},$$

then we say that Ψ has weight (p, t).

The above definition will make more sense once we consider the corresponding action at the Lie algebra level. But, first we note the following consequence of this definition.

Lemma 5.2.6. *If Ψ has weight (k, k) and $r_1(\varphi_1).\Psi(g) = r_2(\varphi_2).\Psi(g) = \Psi(g)$ for* *all* $\varphi_1, \varphi_1 \in \mathbb{R}$ *then*

$$\kappa.\Psi(g) = \Psi(g\kappa) = \Psi(g)\,\det\left(j(\kappa, I)\right)^{-k} \qquad \textit{for all } \kappa \in K.$$

Proof. The result follows form an easy calculation using the definition of weight and the coordinates on K as defined above. □

Now, following Harish-Chandra, as described earlier, in order to understand admissible representations of $\mathrm{Sp}(4, \mathbb{R})$, we shall examine the structure of $(\mathfrak{g}, K)$-modules, where $\mathfrak{g}$ denotes the Lie algebra of $\mathrm{Sp}(4, \mathbb{R})$. Explicitly, $\mathfrak{g} = \{\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M(4, \mathbb{R}) : \ A = -\,{}^tD, \ B = {}^tB, \ C = {}^tC\}$. Let $\mathfrak{k}$ be the Lie algebra of K. A basis of $\mathfrak{k}$ is given by (which could be obtained by, for example, differentiating (5.8) at $\varphi_i = 0$)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{5.9}$$

A convenient basis for the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ is as follows.

$$Z = -i \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, Z' = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, N_+ = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{bmatrix}$$

$$N_- = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{bmatrix}, X_+ = \frac{1}{2} \begin{bmatrix} 1 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad X_- = \frac{1}{2} \begin{bmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{1+} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & i & 0 & -1 \\ i & 0 & -1 & 0 \end{bmatrix}, \ P_{1-} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & -i \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -1 \\ -i & 0 & -1 & 0 \end{bmatrix}, P_{0+} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & -1 \end{bmatrix}$$

$$P_{0-} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & -1 \end{bmatrix}.$$

We note that a basis of $\mathfrak{k}$, as given in (5.9), could also be obtained by calculating the 1-eigenspace of the Cartan involution $\theta: X \mapsto -X^t$ of $\mathfrak{g}$. In any case, it follows from (5.9) that the complexified Lie algebra $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C}$ is spanned by Z, Z', N_+ and N_-. We note the following multiplication table for the Lie algebra $\mathfrak{g}^{\mathbb{C}}$.

	Z	Z'	N_+	N_-	X_+	X_-	P_{1+}	P_{1-}	P_{0+}	P_{0-}
Z	0	0	N_+	$-N_-$	$2X_+$	$-2X_-$	P_{1+}	$-P_{1-}$	0	0
Z'	0	0	$-N_+$	N_-	0	0	P_{1+}	$-P_{1-}$	$2P_{0+}$	$-2P_{0-}$
N_+	$-N_+$	N_+	0	$Z'-Z$	0	$-P_{1-}$	$2X_+$	$-2P_{0-}$	P_{1+}	0
N_-	N_-	$-N_-$	$Z-Z'$	0	$-P_{1+}$	0	$-2P_{0+}$	$2X_-$	0	P_{1-}
X_+	$-2X_+$	0	0	P_{1+}	0	Z	0	N_+	0	0
X_-	$2X_-$	0	P_{1-}	0	$-Z$	0	N_-	0	0	0
P_{1+}	$-P_{1+}$	$-P_{1+}$	$-2X_+$	$2P_{0+}$	0	$-N_-$	0	$Z+Z'$	0	N_+
P_{1-}	P_{1-}	P_{1-}	$2P_{0-}$	$-2X_-$	$-N_+$	0	$-Z-Z'$	0	N_-	0
P_{0+}	0	$-2P_{0+}$	$-P_{1+}$	0	0	0	0	$-N_-$	0	Z'
P_{0-}	0	$2P_{0-}$	0	$-P_{1-}$	0	0	$-N_+$	0	$-Z'$	0

It is clear that Z and Z' span the Cartan subalgebra, say $\mathfrak{h}^{\mathbb{C}}$, of $\mathfrak{g}^{\mathbb{C}}$ as well as $\mathfrak{k}^{\mathbb{C}}$. We define $\mathfrak{g}_\alpha := \{X \in \mathfrak{g}^{\mathbb{C}} : [H, X] = \alpha(H)X, \text{ for all } \alpha \in (\mathfrak{h}^{\mathbb{C}})' := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}^{\mathbb{C}}, \mathbb{C})\}$. Let Δ be the set of all roots, i.e., $\Delta = \{\alpha \in (\mathfrak{h}^{\mathbb{C}})' : \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\}$. We can identify any $\lambda \in (\mathfrak{h}^{\mathbb{C}})'$ with the pair $(\lambda(Z), \lambda(Z'))$. Let $E \cong \mathbb{R}^2$ be the plane that contains all pairs $(\lambda(Z), \lambda(Z'))$ with $\lambda(Z), \lambda(Z') \in \mathbb{R}$. As elements of Δ, i.e., roots, span $(\mathfrak{h}^{\mathbb{C}})'$ we can also say that E is the $\mathbb{R}$-span of roots and it corresponds to the space $(i\mathfrak{b})'$ in Theorem 5.2.4. Further, we say λ is **analytically integral** if $(\lambda(Z), \lambda(Z')) \in \mathbb{Z}^2$. Figure 5.1 shows the analytically integral elements and the root vectors.

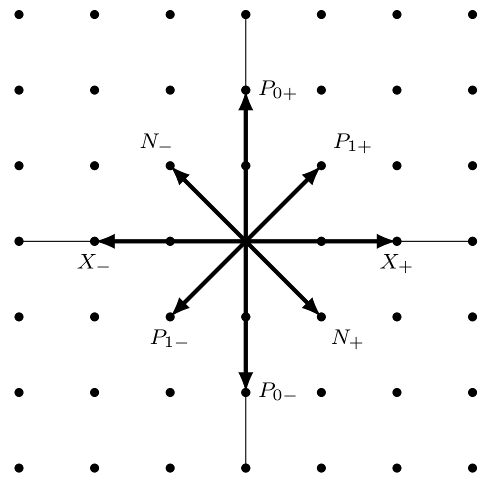

Figure 5.1: The root vectors and the analytically integral elements.

We note that the set of roots is $\Delta = \{(\pm 2, 0), (0, \pm 2), (\pm 1, \pm 1), (\pm 1, \mp 1)\}$ and the set of compact roots is $\Delta_K = \{\pm(1, -1)\}$. Suppose S_α denotes reflection in the hyperplane perpendicular to the root vector α. Then the Weyl group W of this root system is a group with 8 elements generated by $\langle S_\alpha \rangle_{\alpha \in \Delta}$. Let W_K denote the compact Weyl group generated by S_{N_+} = reflections against the hyperplane perpendicular to the compact root N_+. Clearly W_K is a two element group.

K-types and lowest weight representations. We defined the weight of a function on K above (see Def. 5.2.5). By taking the corresponding derived action of the Lie algebra we define a vector v in a representation of $\mathfrak{k}^{\mathbb{C}}$ to be of weight $(p, t) \in \mathbb{Z}^2$ if $Zv = pv$ and $Z'v = tv$. The Lie algebra $\mathfrak{k}^{\mathbb{C}}$ contains a subalgebra isomorphic to $\mathfrak{su}(2)$. In fact, we have the following direct sum decomposition

$$\mathfrak{k}^{\mathbb{C}} = \langle Z - Z', N_+, N_- \rangle \oplus \langle Z + Z' \rangle,$$

with $\langle Z - Z', N_+, N_- \rangle \cong \mathfrak{su}(2)$ and $\langle Z + Z' \rangle$ = span of $Z + Z'$ = the center of $\mathfrak{k}^{\mathbb{C}}$. It follows from the representation theory of $\mathfrak{su}(2)$ that its irreducible representations are characterized by the weight of a highest weight vector, with the highest weight being a non-negative integer. Hence, we get that the isomorphism classes of irreducible representations of K are in one-one correspondence with the set $\{(p,t) \in \mathbb{Z}^2 : \ p \geq t\}$. Here we also note that as infinitesimal characters are defined up to the action of the Weyl group, W_K in this case, only the pairs (p,t) with $p \geq t$ were considered above in characterizing the isomorphism classes of irreducible representations of K. The K-type $V_{(p,t)}$ contains a highest weight vector of weight $\lambda = (p,t)$ annihilated by N_+ and a lowest weight vector of weight (t,p) annihilated by N_-. It is also clear that $V_{(p,t)}$ contains the weight between these two extreme weights with multiplicity one, i.e., each vector with weight in the set $\{(p-i, t+i) : \ 0 \leq i \leq p-t\}$ appears exactly once in $V_{(p,t)}$. It is clear that the dimension of the K-type $V_{(p,t)}$ is $p-t+1$.

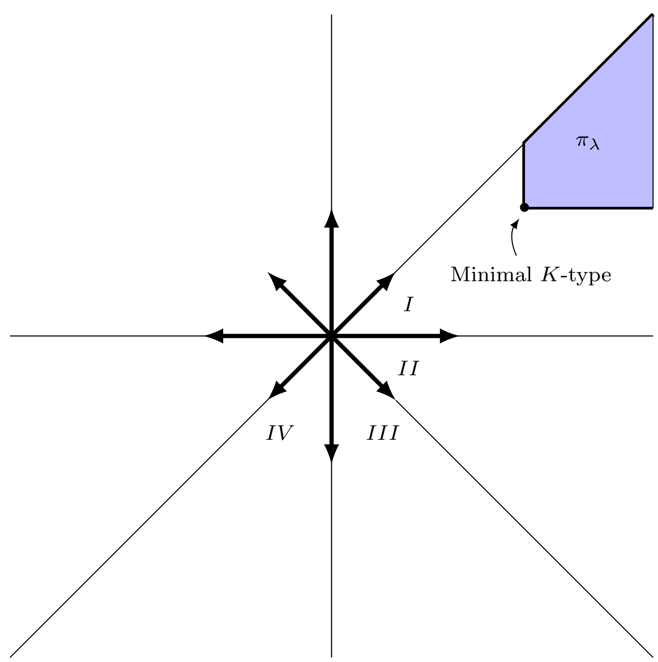

Figure 5.2: The root vectors and the minimal K-type.

It follows from (5.1) that any admissible representation (π_λ, V_K) of $\mathrm{Sp}(4,\mathbb{R})$, i.e., any admissible $(\mathfrak{g}^{\mathbb{C}}, K)$-module, is a direct sum of its constituent K-types, with each K-type occurring with finite multiplicity. We say that V_Λ is the **minimal K-type**

λ in region	Δ^+	δ_G	δ_K	$\Lambda - \lambda = \delta_G - 2\delta_K$
I	$\{(1,1),(2,0),(1,-1),(0,2)\}$	$(2,1)$	$(\frac{1}{2},-\frac{1}{2})$	$(1,2)$
II	$\{(1,1),(2,0),(1,-1),(0,-2)\}$	$(2,-1)$	$(\frac{1}{2},-\frac{1}{2})$	$(1,0)$
III	$\{(-1,-1),(2,0),(1,-1),(0,-2)\}$	$(1,-2)$	$(\frac{1}{2},-\frac{1}{2})$	$(0,-1)$
IV	$\{(-1,-1),(-2,0),(1,-1),(0,-2)\}$	$(-1,-2)$	$(\frac{1}{2},-\frac{1}{2})$	$(-2,-1)$

Table 5.1: The relation between the Harish-Chandra parameter λ and the Blattner parameter Λ for different types of discrete series representations.

contained in V_K if it occurs in V_K with a non-zero multiplicity and $\Lambda = (p,t)$ is closer to the origin than the weight of any other K-type that occurs with non-zero multiplicity in the direct sum decomposition of V_K. In view of Theorem 5.2.4; we calculate and tabulate the relation between the Harish-Chandra parameter λ and the Blattner parameter Λ considering the possibility of λ lying in any one of the regions $\mathrm{I}, \mathrm{II}, \mathrm{III}$ or IV in Figure 5.2.

We note that, according to Theorem 5.2.4, for discrete series to exist λ must be non-singular and $\lambda + \delta_G$ must be integral. Since we see from Table 5.1 above that δ_G is always integral, we assume λ to be analytically integral. We also assume λ to be non-singular. With these assumptions on λ, we obtain the following type of discrete series representations of $\mathrm{Sp}(4,\mathbb{R})$ depending upon the region λ belongs to.

$\boldsymbol{\lambda}$ in region I. We assume that λ lies in the region I. It is evident that π_λ has minimal K-type $\Lambda = \lambda + (1,2)$. The π_λ in this case is said to be in the **holomorphic discrete series**. If $\lambda = (k-1, k-2)$ with $k \geq 3$, then $\Lambda = \lambda + (1,2) = (k,k)$ lies on the main diagonal that is orthogonal to compact roots and so a one-dimensional K-type.

We shall be interested in these discrete series representations in this work, be-

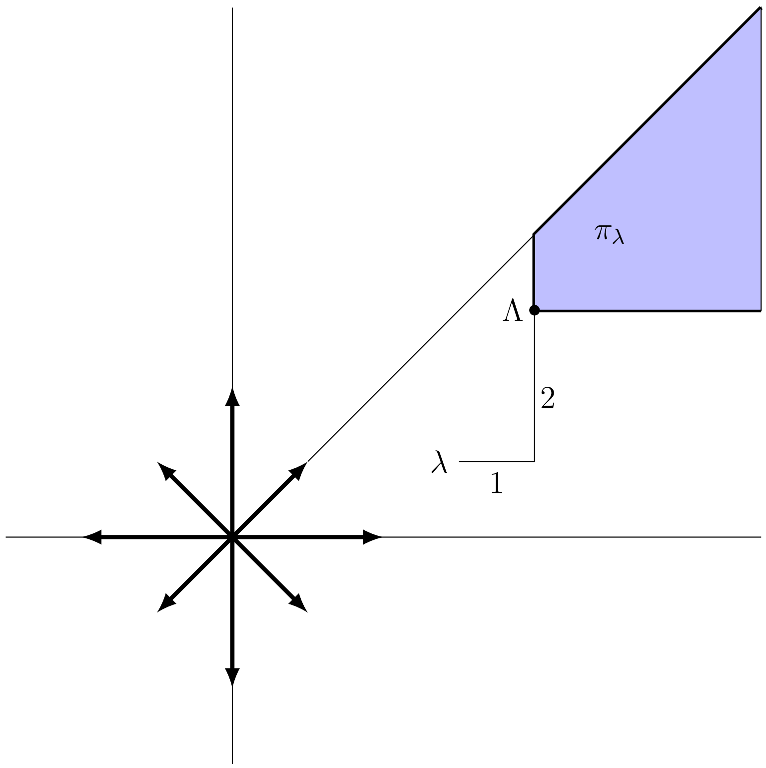

Figure 5.3: Holomorphic discrete series: λ in region I.

cause they arise as the archimedean component of the automorphic representations generated by holomorphic degree 2 Siegel modular forms of weight k.

$\boldsymbol{\lambda}$ in region II. We assume that λ lies in the region II. Then π_λ has minimal K-type $\Lambda = \lambda + (1, 0)$. The π_λ in this case is said to be in the **large discrete series**. These representations are generic representations.

$\boldsymbol{\lambda}$ in region III. We assume that λ lies in the region III. Then π_λ has minimal K-type $\Lambda = \lambda + (0, -1)$. In this case once again we obtain **large discrete series** representations. The picture in this case is symmetric to the one for region II with respect to the diagonal that contains compact roots.

$\boldsymbol{\lambda}$ in region IV. We assume that λ lies in the region IV. Then π_λ has minimal K-type $\Lambda = \lambda + (-2, -1)$. In this case π_λ is known to be in **anti-holomorphic**

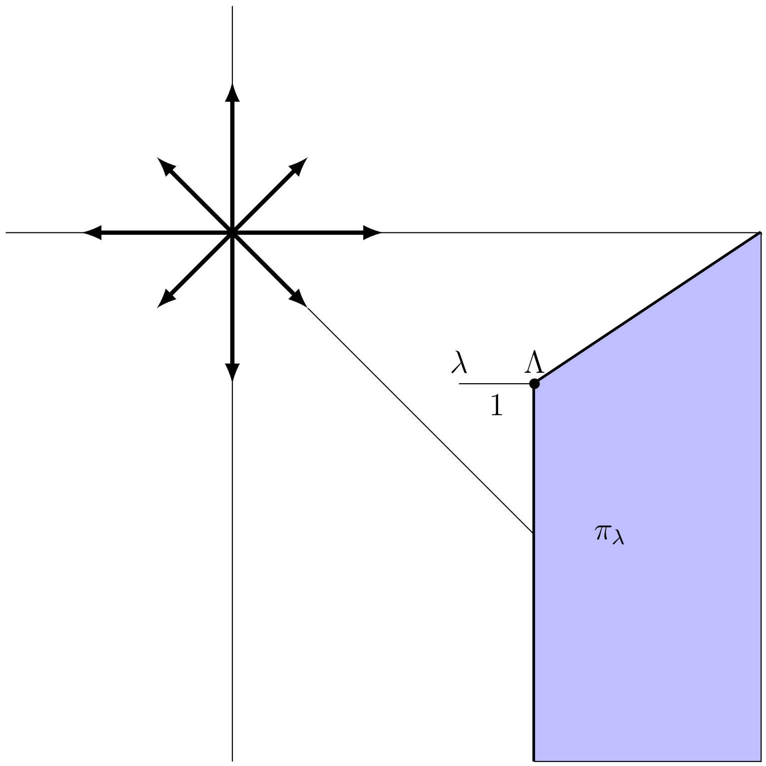

Figure 5.4: Large discrete series: λ in region II.

discrete series representations. The picture in this case is symmetric to the one for region I.

In addition to the discrete series representations described above, there are so called **limits of discrete series representations**, which are obtained in cases when λ is singular. We shall only be interested in the holomorphic discrete series representations and we shall not consider other types of (limits of) discrete series representations in this work.

5.2.3 Representations of $\mathrm{GSp}(4,\mathbb{R}), \mathrm{Sp}(4,\mathbb{R})^{\pm}$

It is clear that any representation of $\mathrm{GSp}(4,\mathbb{R})$ can be restricted to $\mathrm{Sp}(4,\mathbb{R})^{\pm}$, where $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ denotes the subgroup of $\mathrm{GSp}(4,\mathbb{R})$ with multiplier ±1. On the other hand, any representation of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ can be extended to that of $\mathrm{GSp}(4,\mathbb{R})$ by properly defining the action of the center of $\mathrm{GSp}(4,\mathbb{R})$. Since we shall be dealing with representations with trivial central characters, it would be sufficient for us to consider the representations of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$. Next we note that $\mathrm{Sp}(4,\mathbb{R})$ is an index 2 subgroup of $\mathrm{Sp}(4,\mathbb{R})^{\pm}$. More explicitly, $\mathrm{Sp}(4,\mathbb{R})^{\pm} = \mathrm{Sp}(4,\mathbb{R}) \sqcup \gamma\,\mathrm{Sp}(4,\mathbb{R})$ where $\gamma = \mathrm{diag}(1,1,-1,-1)$. We note that if $k \in K$ acts with weight (p,t) on a function Ψ on K then $\gamma k \gamma^{-1}$ acts with weight $(-p,-t)$ on Ψ (c.f., 5.2.5). Suppose (p,t) is in region I. Then the reflection of $(-p,-t)$ in the diagonal orthogonal to compact roots is $(-t,-p)$. Let $\lambda = (p,t)$ and $\lambda' = (-t,-p)$ be analytically integral and non-singular elements of E and let π_λ and $\pi_{\lambda'}$ denote the corresponding discrete series representations. Then we conclude that on inducing the holomorphic discrete series representation π_λ of $\mathrm{Sp}(4,\mathbb{R})$ to $\mathrm{Sp}(4,\mathbb{R})^{\pm}$ we obtain a representation that combines the K-types of π_λ and $\pi_{\lambda'}$.

5.2.4 Holomorphic discrete series representations and holomorphic Siegel modular forms

Following [1] we note the condition for holomorphy of a smooth function f on $\mathbb{H}_n$ that transforms like a modular form. For this we note that associated to f is the adelic function Φ_f defined on $\mathrm{G}(\mathbb{A}) := \mathrm{GSp}(2n,\mathbb{A})$ and given by $\Phi_f(g) = (f|_k g_\infty)(I)$ (c.f., (23) [1]). Here using the strong approximation for G, g is written as $g = g_{\mathbb{Q}} g_\infty k_0$ with $g_{\mathbb{Q}} \in \mathbb{Q}$, $g_\infty \in \mathrm{G}(\mathbb{R})^+$ and $k_0 \in \prod_{p<\infty} \mathrm{G}(\mathbb{Z}_p)$. Then the condition for holomorphy of f could be given in terms of the annihilation of the associated adelic function Φ_f by certain differential operators. Let

$$P_{\mathbb{C}}^{-} = \left\{ \begin{bmatrix} A & -iA \\ -iA & -A \end{bmatrix} \in M(2n,\mathbb{C}) : A = {}^{\mathrm{t}}A \right\}.$$

Then Lemma 7 in [1] says that f is holomorphic if and only if $P_{\mathbb{C}}^{-} \cdot \Phi_f = 0$.

Translating this condition to our situation for $n = 2$ means that in order to obtain holomorphic Siegel modular forms the archimedean distinguished vector must be chosen such that it is annihilated by P_{1-}, P_{0-} and X_{-}.

5.2.5 Klingen parabolic induction and a distinguished vector

Suppose χ is a character of $\mathbb{R}^{\times}$ and π is an admissible representation of $\mathrm{GL}(2,\mathbb{R})$. Then we let $\chi \rtimes \pi$ stand for the representation of $\mathrm{GSp}(4,\mathbb{R})$ obtained by normalized parabolic induction from the representation of the Klingen parabolic subgroup $Q(\mathbb{R})$ given by

$$\begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} \mapsto \chi(t)\,\pi(\begin{bmatrix} a & b \\ c & d \end{bmatrix}).$$

Similarly, let π now be an admissible representation of $\mathrm{SL}(2,\mathbb{R})$ and let χ denote a character of $\mathbb{R}^{\times}$ as earlier. Then we write $\chi \rtimes \pi$ for a representation of $\mathrm{Sp}(4,\mathbb{R})$ induced from the Klingen parabolic subgroup via normalized parabolic induction. There shall not be any confusion, as the context would make it clear for the representation of which group, $\mathrm{GSp}(4,\mathbb{R})$ or $\mathrm{Sp}(4,\mathbb{R})$, the notation $\chi \rtimes \pi$ stands for.

We are interested in the representation $|\cdot|^{k-2} \rtimes |\cdot|^{\frac{-(k-2)}{2}}\pi$ of $\mathrm{GSp}(4,\mathbb{R})$. In view of an earlier discussion (in Subsection 5.2.3), it would be sufficient to consider the representation $|\cdot|^{k-2} \rtimes \pi$ of $\mathrm{Sp}(4,\mathbb{R})$.

Proposition 5.2.7. *Assume $k \geq 4$ to be an even positive integer. Let $\pi = X(k-1,+)$ be a discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ with the lowest K-type k. Then the representation $|\cdot|^{k-2} \rtimes \pi$ of $\mathrm{Sp}(4,\mathbb{R})$ contains a holomorphic discrete series as a subrepresentation with minimal K-type (k,k).*

Proof. The proposition is a special case of Theorem 10.1, (10.2) in [24], with $p = k-1$ and $t = k-2$. This completes the proof. $\square$

For an easy reference we reproduce Theorem 10.1 in [24] below.

Theorem (10.1, [24], Muić). *Assume $p > t > 0$. Then the following sequences are exact:*

$$\begin{aligned} X(p,-t) \oplus X(t,-p) &\hookrightarrow \delta(|\cdot|^{(p-t)/2}\operatorname{sgn}^t, p+t) \rtimes 1 \\ &\twoheadrightarrow \operatorname{Lang}(\delta(|\cdot|^{(p-t)/2}\operatorname{sgn}^t, p+t) \rtimes 1) \end{aligned} \tag{5.10}$$

$$\begin{aligned} X(p,t) \oplus X(p,-t) &\hookrightarrow |\cdot|^t \operatorname{sgn}^t \rtimes X(p,+) \twoheadrightarrow \operatorname{Lang}(|\cdot|^t \operatorname{sgn}^t \rtimes X(p,+)) \\ X(t,-p) \oplus X(-t,-p) &\hookrightarrow |\cdot|^t \operatorname{sgn}^t \rtimes X(p,-) \twoheadrightarrow \operatorname{Lang}(|\cdot|^t \operatorname{sgn}^t \rtimes X(p,-)) \end{aligned} \tag{5.11}$$

Moreover, $X(p,-t)$ and $X(t,-p)$ are large.

Chapter 6

Paramodular Klingen Eisenstein series

6.1 Some useful lemmas

In this chapter we shall give one of our main results, Theorem 6.2.1, which is to obtain a paramodular Klingen Eisenstein series using parabolic induction. But, first we state the following result due to Reefschläger.

Lemma 6.1.1. *Assume $N = \prod_p p^{n_p}$ to be a positive integer. Then,*

$$\mathrm{GSp}(4, \mathbb{Q}) = \bigsqcup_{\gamma \in \mathbb{N}\colon \gamma | N} \mathbb{Q}(\mathbb{Q}) \, \mathrm{L}_\gamma \, \mathrm{K}(N). \tag{6.1}$$

Here

$$\mathrm{L}_\gamma = \begin{bmatrix} 1 & \gamma & & \\ & 1 & & \\ & & 1 & \\ & & -\gamma & 1 \end{bmatrix}.$$

Further, for any non-zero integer x, $\gcd(x, N) = 1$ if and only if

$$\mathbb{Q}(\mathbb{Q}) \, \mathrm{L}_{x\gamma} \, \mathrm{K}(N) = \mathbb{Q}(\mathbb{Q}) \, \mathrm{L}_\gamma \, \mathrm{K}(N). \tag{6.2}$$

Proof. The first assertion is essentially the same as Theorem 1.2 in [29] with a slightly different notation. The second assertion is also clear form the discussion after Theorem 1.2 in [29]. $\square$

Next, we prove the following two lemmas, which will be employed later in proving the main theorem of this chapter.

Lemma 6.1.2. *Assume $N = \prod_p p^{n_p}$ to be a positive integer. Then*

$$\bigcap_{p|N} Q(\mathbb{Q})\mathrm{L}_{p^{n_p}}\mathrm{K}(p^{2n_p}) = Q(\mathbb{Q})\mathrm{L}_N\mathrm{K}(N^2). \tag{6.3}$$

Proof. It is clear that for each prime $p|N$ we have $Q(\mathbb{Q})\mathrm{L}_N\mathrm{K}(N^2) \subseteq Q(\mathbb{Q})\mathrm{L}_N\mathrm{K}(p^{2n_p})$. Further, it follows from Lemma 6.1.1 that $Q(\mathbb{Q})\mathrm{L}_N\mathrm{K}(p^{2n_p}) = Q(\mathbb{Q})\mathrm{L}_{p^{n_p}}\mathrm{K}(p^{2n_p})$. This implies $Q(\mathbb{Q})\mathrm{L}_N\mathrm{K}(N^2) \subseteq \bigcap\limits_{p|N} Q(\mathbb{Q})\mathrm{L}_{p^{n_p}}\mathrm{K}(p^{2n_p})$. Next we prove the other direction. Let $g \in \bigcap\limits_{p|N} Q(\mathbb{Q})\mathrm{L}_{p^{n_p}}\mathrm{K}(p^{2n_p})$. We know, from Lemma 6.1.1, that $\{L_\gamma : \gamma|N^2, \gamma \in \mathbb{N}\}$ constitute a set of representatives for the double coset decomposition of $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\mathrm{K}(N^2)$. Then g must belong to one of these double cosets. Suppose $g \in Q(\mathbb{Q})\,\mathrm{L}_\gamma\,\mathrm{K}(N^2)$ with $\gamma|N^2$. Then $g \in Q(\mathbb{Q})\,\mathrm{L}_\gamma\,\mathrm{K}(p^{2n_p})$ for each prime $p|N$. This means $g \in (Q(\mathbb{Q})\,\mathrm{L}_\gamma\,\mathrm{K}(p^{2n_p})) \cap (Q(\mathbb{Q})\,\mathrm{L}_{p^{n_p}}\,\mathrm{K}(p^{2n_p}))$ for each prime $p|N$. Once again by referring to Lemma 6.1.1, (6.2), for each prime $p|N$, we conclude that $\gamma = p^{n_p}x$ with $\gcd(x,p) = 1$. Therefore, $\gamma = y\prod_{p|N} p^{n_p} = yN$ for some y with $\gcd(y,N) = 1$. But γ also divides N^2, hence $\gamma = N$, and $g \in Q(\mathbb{Q})\mathrm{L}_N\mathrm{K}(N^2)$. This completes the proof of the lemma. $\square$

Lemma 6.1.3.

(a) There exists a bijection between $Q(\mathbb{Q})\backslash Q(\mathbb{Q})\,L_N\,\mathrm{K}(N^2)$ and $\left(L_N^{-1}\,Q(\mathbb{Q})L_N \cap \mathrm{K}(N^2)\right)\backslash\mathrm{K}(N^2)$.

$$Q(\mathbb{Q})\backslash Q(\mathbb{Q})\,L_N\,\mathrm{K}(N^2) \cong \left(L_N^{-1}\,Q(\mathbb{Q})L_N \cap \mathrm{K}(N^2)\right)\backslash\mathrm{K}(N^2). \tag{6.4}$$

(b) The intersection $L_N^{-1}\, Q(\mathbb{Q}) L_N \cap \mathrm{K}(N^2)$ from part (a) is given more explicitly as the set $D(N)$ defined below.

$$D(N) := \pm\{g = L_N^{-1} \begin{bmatrix} 1+N* & & * & \\ & 1 & & \\ N* & & 1+N* & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \mu N^{-1} \\ * & 1 & \mu N^{-1} & kN^{-2} \\ & & 1 & * \\ & & & 1 \end{bmatrix} L_N \in \mathrm{Sp}(4,\mathbb{R})$$
$$|\, *,\ \mu,\ k \in \mathbb{Z},\ \mu \equiv k \mod N\}. \tag{6.5}$$

Proof. The proof of part (a) is trivial and follows from an easy group theoretic consideration. For part (b) let us write $D(N)$ more explicitly. Let $q \in Q(\mathbb{Q})$. Then we can write $q = mu$ where

$$m = \begin{bmatrix} a & & b & \\ & t & & \\ c & & d & \\ & & & t^{-1}\Delta \end{bmatrix}, u = \begin{bmatrix} 1 & & & \mu \\ l & 1 & \mu & \kappa \\ & & 1 & -l \\ & & & 1 \end{bmatrix}$$

with $m, u \in Q(\mathbb{Q})$ and $\Delta = (ad - bc)$.
We want, $(L_N)^{-1}\, q\, L_N \in \mathrm{K}(N^2)$. First of all we note that $\Delta = 1$ from the definition of $\mathrm{K}(N^2)$. Let $M := (L_N)^{-1}\, q\, L_N$. Then we have

$$M = \begin{bmatrix} -lNt+a & -(lNt-a)N-Nt & -\mu Nt+(kNt+bl-a\mu)N+b & -kNt-bl+a\mu \\ lt & lNt+t & -kNt+\mu t & kt \\ c & cN & (dl-c\mu)N+d & -dl+c\mu \\ cN & c(N)^2 & \left(dlN-c\mu N-\frac{1}{t}\right)N+dN & -dlN+c\mu N+\frac{1}{t} \end{bmatrix} \tag{6.6}$$

In the following M_{ij} will indicate the element at i^{th} row and j^{th} column in the matrix M. So, we must have,

$$lt \in \mathbb{Z} \qquad \text{(follows from looking at } M_{21}\text{)}\ , \tag{6.7}$$

$$t \in \mathbb{Z} \qquad \text{(follows from } M_{22} \text{ and (6.7))}, \tag{6.8}$$

$$c \in N\mathbb{Z} \qquad \text{(follows from } M_{41}\text{)}, \tag{6.9}$$

$$c\mu - dl \in \mathbb{Z} \qquad \text{(follows from } (M_{34})\text{)}, \tag{6.10}$$

$$d \in \mathbb{Z} \qquad \text{(follows from } M_{33} \text{ and (6.10))}, \tag{6.11}$$

$$\frac{\Delta}{t} = \frac{1}{t} \in \mathbb{Z} \qquad \text{(follows from } M_{44} \text{ and (6.10))}. \tag{6.12}$$

So, we have $t \in \{-1, 1\}$. We note that, for any $r \in \mathrm{K}(N^2)$,

$$\begin{aligned} &((L_N)^{-1}Q(\mathbb{Q})L_N \cap \mathrm{K}(N^2))\backslash\mathrm{K}(N^2) = \\ &\qquad ((L_N)^{-1}Q(\mathbb{Q})L_N \cap \mathrm{K}(N^2))r\backslash\mathrm{K}(N^2). \end{aligned} \tag{6.13}$$

Let

$$r = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}.$$

$r \in \mathrm{K}(N^2)$. If necessary on multiplying by r, we can assume that

$$t = 1. \tag{6.14}$$

So, now we have,

$$l \in \mathbb{Z} \qquad \text{(follows from (6.14) and (6.7))}, \tag{6.15}$$

$$a \in 1 + N\mathbb{Z} \qquad \text{(follows from } M_{12} \text{ and (6.15))}, \tag{6.16}$$

$$\kappa \in N^{-2}\mathbb{Z} \qquad \text{(follows from } M_{24}\text{)}, \tag{6.17}$$

$$\mu \in N^{-1}\mathbb{Z} \qquad \text{(follows from } M_{23}\text{)}, \tag{6.18}$$

$$b \in \mathbb{Z} \qquad \text{(follows from } M_{14}\text{)}, \tag{6.19}$$

$$\mu \equiv k \mod N \qquad \text{(follows from } M_{23}\text{)}. \tag{6.20}$$

Now the lemma follows. □

6.2 The main result

Now we give the main result of this chapter on a representation theoretic formulation of Klingen Eisenstein series with respect to the paramodular subgroup.

Theorem 6.2.1. *Let S be a finite set of primes and $N = \prod_{p\in S} p^{n_p}$ be a positive integer. Assume χ to be a Dirichlet character modulo N. Let f be an elliptic cusp form of level N, weight k, with $k \geq 6$ an even integer, and character χ, i.e., $f \in S_k^1(\Gamma_0(N), \chi)$. We also assume f to be a newform. Let ϕ be the automorphic form associated with f and let (π, V_π) be the irreducible cuspidal automorphic representation of* $\mathrm{GL}(2,\mathbb{A})$ *generated by ϕ. Moreover, χ could be viewed as a continuous character of ideles, which we also denote by χ. Then there exists a global distinguished vector Φ in the global induced automorphic representation $\chi^{-1}|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$ of* $\mathrm{GSp}(4,\mathbb{A})$*, for $s = k-2$, such that,*

$$\bar{E}(Z) := \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} (\Phi(\gamma b_Z))(1), \tag{6.21}$$

$$(\textit{with } Z = X + iY \in \mathbb{H}_2, b_Z \in B(\mathbb{R}) \textit{ as in } (4.3)),$$

defines an Eisenstein series which is the same as the Klingen Eisenstein series of level N^2 with respect to the paramodular subgroup, defined by

$$E(Z) := \sum_{\gamma \in D(N)\backslash \mathrm{K}(N^2)} f(\mathrm{L}_N \gamma \langle Z \rangle^*) \det(j(\mathrm{L}_N\gamma, Z))^{-k}, \tag{6.22}$$

where,

$$\mathrm{L}_N = \begin{bmatrix} 1 & N & & \\ & 1 & & \\ & & 1 & \\ & & -N & 1 \end{bmatrix} \textit{ and } D(N) = (L_N^{-1}\, Q(\mathbb{Q})\, L_N) \cap \mathrm{K}(N^2).$$

Proof. Let Π denote the automorphic representation $\chi^{-1}|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$ of $\mathrm{GSp}(4,\mathbb{A})$. Π decomposes by the tensor product theorem as

$$\Pi \cong \bigotimes_{p\leq\infty} \Pi_p\,. \tag{6.23}$$

where almost all Π_p are unramified. In the following, for each prime p we shall pick a local distinguished vector $\Phi_p \in \Pi_p$, which would yield a global distinguished vector

$$\Phi = \Phi_\infty \otimes \bigotimes_{p<\infty} \Phi_p\,. \tag{6.24}$$

Similarly, it follows from the tensor product theorem that $\phi = \phi_\infty \otimes \bigotimes_{p<\infty} \phi_p$. Here ϕ is the adelic cusp form associated with $f \in S_1^k(\Gamma_0(N),\chi)$. It is clear that for every finite prime p with $p \nmid N$, ϕ_p is a spherical vector and for each prime p with $p|N$, ϕ_p is a $\mathrm{K}_{\mathrm{p}}^2(p^{n_p})$-invariant vector. Let $\chi = \otimes_p \chi_p$ be the decomposition of χ into local components. Next, we describe our choices for the local distinguished vectors.

Archimedean distinguished vector. For $p=\infty$ we pick a distinguished vector Φ_∞ such that it is of the minimal K-type (k,k). It was shown in Proposition 5.2.7 that such a vector exists in Π_∞. More explicitly we define Φ_∞ as follows

$$\Phi_\infty(h_\infty k_\infty) := \det(j(k_\infty,I))^{-k}|t^2(ad-bc)^{-1}|\,|t|^s\,|ad-bc|^{-\frac{s}{2}}\pi_\infty(\begin{bmatrix} a & b \\ c & d\end{bmatrix})\,\phi_\infty \tag{6.25}$$

where

$$h_\infty = \begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc)\end{bmatrix} \in Q(\mathbb{R})$$

and $k_\infty \in \mathrm{K}_1$, the maximal standard compact subgroup of $\mathrm{GSp}(4,\mathbb{R})$. It can be checked that Φ_∞ is well defined.

Unramified non-archimedean distinguished vectors. For every prime q such that $q \nmid N$, we pick unramified local distinguished vectors such that,

$$\Phi_q(g) = \Phi_q(h_q k_q) = |t^2(ad-bc)^{-1}|_q \, |t|_q^s \, |ad-bc|_q^{-\frac{s}{2}} \pi_q(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\phi_q, \tag{6.26}$$

where using the Iwasawa decomposition $g \in \mathrm{GSp}(4, \mathbb{Q}_q)$ is written as $g = h_q k_q$ with

$$h_q = \begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} \in Q(\mathbb{Q}_q)$$

and $k_q \in \mathrm{GSp}(4, \mathbb{Z}_q)$. It can be checked that Φ_q is well defined.

Ramified non-archimedean distinguished vector. For each finite prime p such that $p|N$ we select a local new form, i.e., a paramodular vector of level p^{2n_p}, as distinguished vector. More explicitly, we pick our distinguished vector Φ_p as a $\mathrm{K}_p(p^{2n_p})$-invariant vector. The existence of such a vector and that it is supported only on $Q(\mathbb{Q}_p)\,\mathrm{L}_{n_p}\,\mathrm{K}_p(p^{2n_p})$ follows from the proof of the Theorem 5.4.2 in [30]. So, our distinguished vector Φ_p is zero on all double cosets other than $Q(\mathbb{Q}_p)\,\mathrm{L}_{n_p}\,\mathrm{K}_p(p^{2n_p})$ and on $Q(\mathbb{Q}_p)\,\mathrm{L}_{n_p}\,\mathrm{K}_p(p^{2n_p})$ it is given by

$$\Phi_p\left(\begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} \mathrm{L}_{n_p}\,\kappa\right) = |t^2(ad-bc)^{-1}|_p \, |t|_p^s \, \chi_p^{-1}(t) |ad-bc|_p^{-\frac{s}{2}} \pi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\phi_p \tag{6.27}$$

where, $\kappa \in \mathrm{K}_p(p^{2n_p})$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Q}_p)$ and $\phi_p \in \nu^{-s/2}\pi_p$ is such that

$$\nu^{-\frac{s}{2}}\pi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\phi_p = \omega_{\nu^{-\frac{s}{2}}\pi_p}(a)\phi_p \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^{n_p}\mathbb{Z}_p & \mathbb{Z}_p^\times \end{bmatrix}.$$

We note that the hypothesis of the Theorem 5.4.2 in [30] requires that, the representation $\chi_p^{-1}\nu^{k-2} \rtimes \nu^{-(k-2)/2}\pi_p$ should have trivial central character. This condition is satisfied as we know that in the passage from $f \in S_k^1(\Gamma_0(N), \chi_d)$ to its associated adelic cusp form, the central character ω of the adelic cusp form is given by the adelization of χ (see section 3.6 in [5]).

Now the Klingen Eisenstein series coming from the global distinguished vector Φ is obtained as

$$\begin{aligned}
\bar{E}(Z) &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \Phi(\gamma b_Z)(1) \\
&= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \left(\Phi_\infty(\gamma b_Z) \bigotimes_{p|N} \Phi_p(\gamma) \bigotimes_{q<\infty,\, q\nmid N} \Phi_q(\gamma) \right)(1) \\
&= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash Q(\mathbb{Q})\mathrm{L}_N \mathrm{K}(N^2)} \left(\Phi_\infty(\gamma b_Z) \bigotimes_{p|N} \Phi_p(\gamma) \bigotimes_{q<\infty,\, q\nmid N} \Phi_q(\gamma) \right)(1) \\
&= \det(Y)^{-k/2} \sum_{\gamma \in \bar{D}(N)} \left(\Phi_\infty(\mathrm{L}_N \gamma b_Z) \bigotimes_{p|N} \Phi_p(\mathrm{L}_N\gamma) \bigotimes_{q<\infty,\, q\nmid N} \Phi_q(\mathrm{L}_N\gamma) \right)(1),
\end{aligned}$$

where, $\bar{D}(N) = \left(\mathrm{L}_N^{-1}\, Q(\mathbb{Q})\, \mathrm{L}_N \cap \mathrm{K}(N^2)\right) \backslash \mathrm{K}(N^2)$.

In the above calculation the third equality follows from the Lemma 6.1.1, the fact that Φ_p is supported only on $Q(\mathbb{Q}_p)\mathrm{L}_N\mathrm{K}_p(p^{2N})$ (see (6.27)) and the Lemma 6.1.2. We have used Lemma 6.1.3 for the last equality. If $q \nmid N$ then $\Phi_q(\mathrm{L}_N\gamma) = \Phi_q(1) = \phi_q$, as Φ_q is unramified and $\mathrm{L}_N\gamma \in \mathrm{GSp}(4, \mathbb{Z}_q)$, the maximal compact subgroup of $\mathrm{GSp}(4, \mathbb{Q}_q)$. Also from the definition of Φ_p we get that $\Phi_p(\mathrm{L}_N\gamma) = \phi_p$. Therefore, we obtain,

$$\begin{aligned}
\bar{E}(Z) &= \det(Y)^{-k/2} \sum_{\gamma \in \bar{D}(N)} \left(\Phi_\infty(\mathrm{L}_N \gamma b_Z) \bigotimes_{p<\infty} \phi_p \right)(1) \\
&= \det(Y)^{-k/2} \sum_{\gamma \in \bar{D}(N)} \left(\Phi_\infty(b_{\tilde{Z}} \kappa_\infty) \bigotimes_{p<\infty} \phi_p \right)(1),
\end{aligned}$$

where $b_{\tilde{Z}}$ is as in (4.17), $b_{\tilde{Z}}\kappa_\infty = \mathrm{L}_N\gamma b_Z$ and $\kappa_\infty \in \mathrm{K}_1$. Then,

$$\begin{aligned}\Phi_\infty(b_{\tilde{Z}}\kappa_\infty) &= \pi_\infty\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & \\ & \tilde{b}^{-1}\end{bmatrix}\right)\Phi_\infty(1)\det(j(\kappa_\infty, I))^{-k}|\tilde{a}|^k\\ &= \det(j(\kappa_\infty, I))^{-k}|\tilde{a}|^k\pi_\infty\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & \\ & \tilde{b}^{-1}\end{bmatrix}\right)\phi_\infty.\end{aligned}$$

So, we get

$$\begin{aligned}\bar{E}(Z) &= \det(Y)^{-k/2}\sum_{\gamma\in\bar{D}(N)}\phi\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & \\ & \tilde{b}^{-1}\end{bmatrix}\right)\det(j(\kappa_\infty, I))^{-k}|\tilde{a}|^k\\ &=^{4.14} \det(Y)^{-k/2}\sum_{\gamma\in\bar{D}(N)}\phi\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & \\ & \tilde{b}^{-1}\end{bmatrix}\right)\\ &\qquad\qquad \left(\sqrt{\frac{\det\tilde{Y}}{\det Y}}\det(j(\mathrm{L}_N\,\gamma, Z))\right)^{-k}|\tilde{a}|^k\\ &=^{4.4} \sum_{\gamma\in\bar{D}(N)}\phi\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{b} & \\ & \tilde{b}^{-1}\end{bmatrix}\right)\left(\sqrt{\det\tilde{Y}}\det(j(\mathrm{L}_N\,\gamma, Z))\right)^{-k}\left(\frac{\det(\tilde{Y})}{\tilde{y}}\right)^{k/2}\\ &= \sum_{\gamma\in\bar{D}(N)}\phi\left(\begin{bmatrix}1 & \tilde{x}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{\tilde{y}} & \\ & (\sqrt{\tilde{y}})^{-1}\end{bmatrix}\right)(\tilde{y})^{-k/2}\det(j(\mathrm{L}_N\,\gamma, Z))^{-k}\\ &= \sum_{\gamma\in\bar{D}(N)} f(\tilde{x}+i\tilde{y})\det(j(\mathrm{L}_N\,\gamma, Z))^{-k}\\ &= \sum_{\gamma\in\bar{D}(N)} f(\mathrm{L}_N\gamma\langle Z\rangle^*)\det(j(\mathrm{L}_N\,\gamma, Z))^{-k}.\end{aligned}$$

Thus we obtain the desired Klingen Eisenstein series. □

Chapter 7

Klingen Eisenstein series for $\Gamma_0(N)$

In this chapter we will obtain a precise representation theoretic formulation of Klingen Eisenstein series with respect to the congruence subgroups $\Gamma_0(N)$. In the following sections we will make use of certain double coset decompositions which will only be proven later in the next chapter.

To begin with we reformulate Proposition 8.3.1 using the classical version of GSp(4) and note the result in the following.

Proposition 7.0.1. *Assume $n \geq 1$. A complete and minimal system of representatives for the double cosets $Q(\mathbb{Q}_p)\backslash \mathrm{GSp}(4, \mathbb{Q}_p)/\mathrm{Si}(p^n)$ is given by*

$$1, \qquad s_1 s_2, \qquad h_r = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & p^r & & 1 \end{bmatrix}, \ 1 \leq r \leq n-1,$$

$$g_s = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & p^s & 1 & \\ p^s & & & 1 \end{bmatrix}, \ 1 \leq s \leq n-1,$$

$$h_{(r,s)} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & p^s & 1 & \\ p^s & p^r & & 1 \end{bmatrix}, \ 1 \leq s, r \leq n-1, \ s < r < 2s.$$

7.1 Existence of well-defined local vectors

Let $f : \mathrm{GSp}(4, \mathbb{Q}_p) \to V_{\pi_p}$ be a smooth function in $\chi_p\nu^s \rtimes \nu^{-s/2}\pi_p$, where π_p is an irreducible, admissible representation of $\mathrm{GL}(2, \mathbb{Q}_p)$ and $\chi_p = \omega_{\pi_p}^{-1}$ is the inverse of the central character of π_p. Let $a(\pi_p)$ be the conductor of π_p. Assume n to be a non-negative integer. Let f be $\mathrm{Si}(p^n)$ invariant. Then it is clear that f is determined by its values on a set of double-coset representatives of $Q(\mathbb{Q}_p)\backslash\mathrm{GSp}(4, F)/\mathrm{Si}(p^n)$, such as the one given by Proposition 7.0.1. Let h denote any of these representatives. We further assume that f is supported only on the double coset $Q(\mathbb{Q}_p)h\mathrm{Si}(p^n)$. Let $f(h) = v_h$. Then for f to be well defined it must satisfy the following condition,

$$|t^2 \det(A)^{-1}|_p\, \chi_p(t)|t|_p^s|\det(A)|_p^{-s/2}\pi_p(A)\, v_h = v_h \tag{7.1}$$
$$\text{for all } g = \begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}\det(A) \end{bmatrix} \in Q(\mathbb{Q}_p) \cap h\,\mathrm{Si}(p^n)h^{-1}.$$

Suppose $q \in Q(\mathbb{Q}_p) \cap h\,\mathrm{Si}(p^n)h^{-1}$. Then we must have $h^{-1}qh \in \mathrm{Si}(p^n)$. In the following we shall explore if (7.1) is satisfied and f is well defined for various choices of h. We will take a general $q \in Q(\mathbb{Q}_p)$ as having the form $q = mu$ with $m = \begin{bmatrix} a & & b & \\ & t & & \\ c & & d & \\ & & & \frac{1}{t} \end{bmatrix}$ and $u = \begin{bmatrix} 1 & & & \mu \\ l & 1 & \mu & k \\ & & 1 & -l \\ & & & 1 \end{bmatrix}$.

<u>**h = 1**</u>. To check the condition (7.1) we take $g = \begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}\det(A) \end{bmatrix} \in Q(\mathbb{Q}_p) \cap \mathrm{Si}(p^n)$. Then it is clear from the definition of $\mathrm{Si}(p^n)$ that $\det(g) = 1$. It also follows from the definition of $\mathrm{Si}(p^n)$ that $b \in \mathbb{Z}_p$, $c \in p^n\mathbb{Z}_p$. Also from the compactness of $\mathrm{Si}(p^n)$ it follows that $|t|_p = 1$, so t, a, $d \in \mathbb{Z}_p^\times$. Therefore the condition (7.1) is satisfied for all matrices $g \in Q(\mathbb{Q}_p) \cap \mathrm{Si}(p^n)$ if and only if χ_p is unramified and v_1 is

such that

$$\pi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix})v_1 = v_1 \qquad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{bmatrix}. \tag{7.2}$$

It is clear that if $a(\pi_p) > n$ then such a function f can not exist. If $a(\pi_p) = n$ then on taking v_1 to be a local newform we get that the dimension of functions supported on the double coset $Q(\mathbb{Q}_p)1\mathrm{Si}(p^n)$ is 1. This will be the case of our interest.

<u>$\mathbf{h = s_1 s_2}$.</u> For $h = s_1 s_2$ we have

$$s_2^{-1}s_1^{-1}mus_1s_2 = \begin{bmatrix} \frac{1}{t} & & & \\ bl - a\mu & a & & b \\ -kt & lt & t & \mu t \\ dl - c\mu & c & & d \end{bmatrix}.$$

It follows from the definition of $\mathrm{Si}(p^n)$ that $b \in \mathbb{Z}_p$, $c \in p^n\mathbb{Z}_p$. Also as noted earlier, $\det(m) = \Delta^2 = (ad-bc)^2 \in \mathbb{Z}_p^\times$ and $|t|_p = 1$, so t, a, $d \in \mathbb{Z}_p^\times$. Therefore the condition (7.1) is satisfied for all matrices $g \in Q(\mathbb{Q}_p) \cap s_1s_2\,\mathrm{Si}(p^n)s_2^{-1}s_1^{-1}$ if and only if χ_p is unramified and $v_{s_1s_2}$ is such that

$$\pi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix})v_{s_1s_2} = v_{s_1s_2} \qquad \text{for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^n\mathbb{Z}_p & \mathbb{Z}_p^\times \end{bmatrix}. \tag{7.3}$$

That the above condition is necessary follows by noting that given $t \in \mathbb{Z}_p^\times$ and $\left[\begin{smallmatrix} a & b \\ c & b \end{smallmatrix}\right]$ as in (7.3)

$$s_2^{-1}s_1^{-1}us_1s_2 = \begin{bmatrix} \frac{1}{t} & & & \\ & a & & b \\ & & t & \\ & c & & d \end{bmatrix} \in \mathrm{Si}(p^n).$$

Once again it is clear that if $a(\pi_p) > n$ then such a function f can not exist. If $a(\pi_p) = n$ then on taking $v_{s_1s_2}$ to be a local newform we see that the dimension of functions supported on the double coset $Q(\mathbb{Q}_p)s_1s_2\mathrm{Si}(p^n)$ is 1.

7.2 Intersection lemmas

Next, we prove the following lemmas which will be useful in proving the main results of this chapter.

Lemma 7.2.1. *Let $N = \prod_p p^{n_p}$ be a positive integer. Then,*

(i) $\bigcap\limits_{p|N} Q(\mathbb{Q})\,1\,\Gamma_0^4(p^{n_p}) = Q(\mathbb{Q})\,1\,\Gamma_0^4(N)$.

(ii) $\bigcap\limits_{p|N} Q(\mathbb{Q})\,s_1s_2\,\Gamma_0^4(p^{n_p}) = Q(\mathbb{Q})\,s_1s_2\,\Gamma_0^4(N)$.

(iii) $\bigcap\limits_{p|N} Q(\mathbb{Q})\,h_p\,\Gamma_0^4(p^{2n_p}) = Q(\mathbb{Q})\,h\,\Gamma_0^4(N^2)$. *where*

$$h_p = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & p^{n_p} & 1 & \\ p^{n_p} & & & 1 \end{bmatrix} \text{ and } h = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & N & 1 & \\ N & & & 1 \end{bmatrix}.$$

Proof. We shall prove only the first part of the lemma, i.e., we shall show that

$$\bigcap_{p|N} Q(\mathbb{Q})\,1\,\Gamma_0^4(p^{n_p}) = Q(\mathbb{Q})\,1\,\Gamma_0^4(N).$$

The remaining parts are similar. The implication $\bigcap\limits_{p|N} Q(\mathbb{Q})\,1\,\Gamma_0^4(p^{n_p}) \supseteq Q(\mathbb{Q})\,1\,\Gamma_0^4(N)$ is clear. Therefore, we need to prove the other direction. Let $g \in \bigcap\limits_{p|N} Q(\mathbb{Q})\,1\,\Gamma_0^4(p^{n_p})$. Consider g as an element of $\mathrm{GSp}(4,\mathbb{Q})$, then it must lie in one of the double cosets having representatives as listed in Theorem 8.2.2. If $g \in Q(\mathbb{Q})\,1\,\Gamma_0^4(N)$ then we are done. Hence, suppose $g \in Q(\mathbb{Q})\,s_1s_2\,\Gamma_0^4(N)$. Then for any $p|N$ we get that $g \in Q(\mathbb{Q})\,s_1s_2\,\Gamma_0^4\,(p^{n_p})$ but this contradicts that $g \in Q(\mathbb{Q})\,1\,\Gamma_0^4(p^{n_p})$. Next, suppose $g \in Q(\mathbb{Q})\,g_1(\gamma,x)\,\Gamma_0^4(N)$ where $g_1(\gamma,x)$ is defined as in Theorem 8.2.2. But this means

$$g \in (Q(\mathbb{Q})\,g_1(\gamma,x)\,\Gamma_0^4(p^{n_p})) \cap (Q(\mathbb{Q})\,1\,\Gamma_0^4\,1\,(p^{n_p}))$$

for each prime $p|N$. This forces $p^{n_p}|\gamma$ for each $p|N$. Therefor $N|\gamma$ but then it contradicts that $\gamma < N$. Therefore, $g \notin Q(\mathbb{Q})\, g_1(\gamma, x)\, \Gamma_0^4(N)$. A similar argument shows that g does not lie in any of the double cosets represented by the remaining representatives $g_2(\eta)$ and $g_3(\gamma, \delta, y)$. Therefore, we conclude that g must lie in $g \in Q(\mathbb{Q})\, 1\, \Gamma_0^4(N)$. This completes the proof of the lemma.

□

Lemma 7.2.2. *There exists a bijection between* $Q(\mathbb{Q})\backslash Q(\mathbb{Q})\, h\, \Gamma_0^4(N)$ *and* $(h^{-1}\, Q(\mathbb{Q})\, h \cap \Gamma_0^4(N))\, \backslash \Gamma_0^4(N)$, *where* h *is any one of the representatives listed in Theorem 8.2.2.*

$$Q(\mathbb{Q})\backslash Q(\mathbb{Q})\, h\, \Gamma_0^4(N) \cong \left(h^{-1}\, Q(\mathbb{Q})\, h \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N) \tag{7.4}$$

Proof. The proof is trivial. □

The following lemma gives a more explicit description of $h^{-1}\, Q(\mathbb{Q})\, h \cap \Gamma_0^4(N)$ for $h = s_1 s_2$.

Lemma 7.2.3.

$$\begin{aligned} &(s_1 s_2)^{-1}\, Q(\mathbb{Q})\, s_1 s_2 \cap \Gamma_0^4(N) = \\ &\qquad \pm \left\{ \begin{bmatrix} & & & -1 \\ a & & b & \\ & 1 & & \\ c & & d & \end{bmatrix} \begin{bmatrix} -\mu & 1 & & \\ -k & l & 1 & \mu \\ l & & & 1 \\ -1 & & & \end{bmatrix} \cap \mathrm{Sp}(4, \mathbb{R}) \,\middle|\, c,\ l,\ k \in N\mathbb{Z},\ a,\ b,\ d,\ \mu \in \mathbb{Z} \right\} \end{aligned} \tag{7.5}$$

Proof. Let $q \in Q(\mathbb{Q})$. Then we can write $q = mu$ where

$$m = \begin{bmatrix} a & & b & \\ & t & & \\ c & & d & \\ & & & t^{-1}\Delta \end{bmatrix}, u = \begin{bmatrix} 1 & & & \mu \\ l & 1 & \mu & \kappa \\ & & 1 & -l \\ & & & 1 \end{bmatrix}$$

with $m, u \in Q(\mathbb{Q})$ and $\Delta = ad - bc$.

If $(s_1s_2)^{-1}\,q\,s_1s_2 \in \Gamma_0^4(N)$. Then $\Delta = 1$ from the definition of $\Gamma_0^4(N)$. We also have,

$$(s_1s_2)^{-1}\,q\,s_1s_2 = \begin{bmatrix} & & & -1 \\ a & & b & \\ & 1 & & \\ c & & d & \end{bmatrix} \begin{bmatrix} -\mu & 1 & & \\ -k & l & 1 & \mu \\ l & & & 1 \\ -1 & & & \end{bmatrix} = \begin{bmatrix} & \frac{1}{t} & & \\ bl - a\mu & a & & b \\ -kt & lt & t & \mu t \\ dl - c\mu & c & & d \end{bmatrix} \in \Gamma_0^4(N). \quad (7.6)$$

It follows that $t = \pm 1$. In fact, we can assume $t = 1$ by multiplying with an appropriate element in $\Gamma_0^4(N)$. We also get $c,\, l,\, k \in N\mathbb{Z}$ and $a,\, b,\, d,\, \mu \in \mathbb{Z}$. The result follows. $\square$

7.3 Main results

Now we are ready to prove the following result, which shows the existence of a Klingen Eisenstein series of level N with respect to the subgroup $\Gamma_0^4(N)$.

Theorem 7.3.1. *Let S be a finite set of primes and $N = \prod_{p\in S} p^{n_p}$ be a positive integer. Let f be an elliptic cusp form of level N, weight k, i.e., $f \in S_k^1(\Gamma_0^2(N))$. We also assume f to be a newform. Let ϕ be the automorphic form associated with f and let (π, V_π) be the irreducible cuspidal automorphic representation of* $\mathrm{GL}(2,\mathbb{A})$ *generated by ϕ. Then there exists a global distinguished vector Φ in the global induced automorphic representation $|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$ of* $\mathrm{GSp}(4,\mathbb{A})$, *for $s = k-2$, such that,*

$$\bar{E}(Z) := \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} (\Phi(\gamma b_Z))(1), \quad (7.7)$$

(with $Z = X + iY \in \mathbb{H}_2, b_Z$ as in (4.5)*)*

defines an Eisenstein series which is same as the Klingen Eisenstein series of level N with respect to $\Gamma_0^4(N)$, defined by

$$E(Z) := \sum_{\gamma \in \left(Q(\mathbb{Q})\cap\Gamma_0^4(N)\right)\backslash\, \Gamma_0^4(N)} f(\gamma\langle Z\rangle^*) \det(j(\gamma, Z))^{-k}, \quad (7.8)$$

Proof. Let Π denote the automorphic representation $|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}}\pi$ of $\mathrm{GSp}(4,\mathbb{A})$. We know from the tensor product theorem that

$$\Pi \cong \bigotimes_{p\leq\infty} \Pi_p \tag{7.9}$$

where almost all Π_p are unramified. In the following, for each prime p we shall pick a local distinguished vector $\Phi_p \in \Pi_p$, which would yield a global distinguished vector

$$\Phi = \Phi_\infty \otimes \bigotimes_{p<\infty} \Phi_p\,. \tag{7.10}$$

Similarly, it follows from the tensor product theorem that

$$\phi \cong \phi_\infty \otimes \bigotimes_{p<\infty} \phi_p.$$

Here, as ϕ is the adelic cusp form associated with $f \in S_k^1(\Gamma_0^2(N))$, it is clear that for every finite prime p with $p \nmid N$, ϕ_p is a spherical vector and for each prime p with $p|N$, ϕ_p is a

$$\mathrm{K}_\mathrm{p}^2(p^{n_p}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Z}_p)\colon c \in p^{n_p}\mathbb{Z}_p,\, d \in 1 + p^{n_p}\mathbb{Z}_p \right\}$$

invariant vector. Next, we describe our choices for the local distinguished vectors.

Archimedean distinguished vector

For $p=\infty$ we pick a distinguished vector Φ_∞ such that it is of the minimal K-type (k,k). It was shown in Proposition 5.2.7 that such a vector exists in Π_∞. A more explicit description of Φ_∞ is given in (6.25).

Unramified non-archimedean distinguished vectors

For all primes q such that $q \nmid N$, we pick unramified local distinguished vectors such that,

$$\Phi_q(1) := \phi_q\,. \tag{7.11}$$

This means we have

$$\Phi_q(g) = \Phi_q(h_q k_q) = |t^2(ad-bc)^{-1}|_q\,|t|_q^s\,|ad-bc|_q^{-\frac{s}{2}}\pi_q(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\phi_q, \tag{7.12}$$

where using the Iwasawa decomposition $g \in \mathrm{GSp}(4,\mathbb{Q}_q)$ is written as $g = h_q k_q$ with

$$h_q = \begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} \in Q(\mathbb{Q}_q)$$

and $k_q \in \mathrm{GSp}(4,\mathbb{Z}_q)$. It is easy to check that Φ_q is well defined.

Ramified non-archimedean distinguished vector

For each finite prime p such that $p|N$ we select a $\Gamma_0^4(p^{n_p})$ invariant vector as distinguished vector. We note that the existence of such a vector that is supported only on $Q(\mathbb{Q}_p)1\Gamma_0^4(p^{2n_p})$ follows from (7.2) and the discussion preceding that. Therefore, our distinguished vector Φ_p is zero on all double cosets other than $Q(\mathbb{Q}_p)1\Gamma_0^4(p^{2n_p})$ and on $Q(\mathbb{Q}_p)1\Gamma_0^4(p^{2n_p})$ it is given by

$$\begin{aligned}\Phi_p\left(\begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} 1\,\kappa\right) =& |t^2(ad-bc)^{-1}|_p\,|t|_p^s \\ & |ad-bc|_p^{-\frac{s}{2}}\pi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\phi_p\end{aligned} \tag{7.13}$$

where, $\kappa \in \mathrm{Si}(p^n), \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2,\mathbb{Q}_p)$ and $\phi_p \in \pi_p$ is such that

$$\pi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\phi_p = \phi_p \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^{n_p}\mathbb{Z}_p & \mathbb{Z}_p^\times \end{bmatrix}.$$

Now the Klingen Eisenstein series coming from the global distinguished vector Φ is obtained as

$$\begin{aligned}
\bar{E}(Z) &= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \Phi(\gamma b_Z)(1) \\
&= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})} \left(\Phi_\infty(\gamma b_Z) \bigotimes_{p|N} \Phi_p(\gamma) \bigotimes_{q<\infty,\, q\nmid N} \Phi_q(\gamma) \right)(1) \\
&= \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q})\backslash \left(\underset{p|N}{\cap} Q(\mathbb{Q})\Gamma_0^4(p^{n_p}) \right)} \left(\Phi_\infty(\gamma b_Z) \bigotimes_{p|N} \Phi_p(\gamma) \bigotimes_{q<\infty,\, q\nmid N} \Phi_q(\gamma) \right)(1) \\
&= \det(Y)^{-k/2} \sum_{\gamma \in \left(Q(\mathbb{Q})\cap\Gamma_0^4(N) \right)\backslash \Gamma_0^4(N)} \left(\Phi_\infty(\gamma b_Z) \bigotimes_{p|N} \Phi_p(\gamma) \bigotimes_{q<\infty,\, q\nmid N} \Phi_q(\gamma) \right)(1)
\end{aligned}$$

In the above calculation the third equality needs a little explanation. In fact, only those $\gamma \in \mathrm{GSp}(4,\mathbb{Q})$ contribute to the sum, which, for each prime $p|N$, considered as an element of $\mathrm{GSp}(4,\mathbb{Q}_p)$ belong to the double coset $Q(\mathbb{Q}_p)\,1\,\mathrm{Si}(p^{n_p})$; as Φ_p is supported only on $Q(\mathbb{Q}_p)\,1\,\mathrm{Si}(p^{n_p})$ (see (7.13)). It then follows from the double coset decompositions of $Q(\mathbb{Q}_p)\backslash\mathrm{GSp}(4,\mathbb{Q}_p)/\mathrm{Si}(p^{n_p})$ (see Proposition 7.0.1) and $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0^4(p^{n_p})$ (see Theorem 8.1.10) that any such $\gamma \in \mathrm{GSp}(4,\mathbb{Q})$, contributing non-trivially to the sum, must belong to $\underset{p|N}{\cap} Q(\mathbb{Q})\Gamma_0^4(p^{n_p})$.

If $q \nmid N$ then $\Phi_q(\gamma) = \Phi_q(1) = \phi_q$, as Φ_q is unramified and $\gamma \in \mathrm{GSp}(4,\mathbb{Z}_q)$ the maximal compact subgroup of $\mathrm{GSp}(4,\mathbb{Q}_q)$. Also from the definition of Φ_p, we get that $\Phi_p(\gamma) = \phi_p$. Therefore, we obtain

$$\bar{E}(Z) = \det(Y)^{-k/2} \sum_{\gamma \in \left(Q(\mathbb{Q})\cap\Gamma_0^4(N) \right)\backslash \Gamma_0^4(N)} \left(\Phi_\infty(\gamma b_Z) \bigotimes_{p<\infty} \phi_p \right)(1)$$

$$= \det(Y)^{-k/2} \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} \left(\Phi_\infty(b_{\tilde{Z}} \kappa_\infty) \bigotimes_{p<\infty} \phi_p \right)(1),$$

where $b_{\tilde{Z}}$ is as in (4.17), $b_{\tilde{Z}}\kappa_\infty = \gamma b_Z$ and $\kappa_\infty \in \mathrm{K}_1$. Then

$$\begin{aligned}
&\Phi_\infty(b_{\tilde{Z}}\kappa_\infty)(1) \\
&= \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} \left(\pi_\infty \left(\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{b} & \\ & \tilde{b}^{-1} \end{bmatrix} \right) \Phi_\infty(1) \right)(1) \det(j(\kappa_\infty, I))^{-k} |\tilde{a}|^k \\
&= \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} \det(j(\kappa_\infty, I))^{-k} |\tilde{a}|^k \left(\pi_\infty \left(\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{b} & \\ & \tilde{b}^{-1} \end{bmatrix} \right) \phi_\infty \right)(1) \\
&= \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} \det(j(\kappa_\infty, I))^{-k} |\tilde{a}|^k \, \phi_\infty \left(\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{b} & \\ & \tilde{b}^{-1} \end{bmatrix} \right).
\end{aligned}$$

So, we get

$$\begin{aligned}
\bar{E}(Z) &= \det(Y)^{-k/2} \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} \phi \left(\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{b} & \\ & \tilde{b}^{-1} \end{bmatrix} \right) \det(j(\kappa_\infty, I))^{-k} |\tilde{a}|^k \\
&=^{(4.14)} \det(Y)^{-k/2} \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} \phi \left(\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{b} & \\ & \tilde{b}^{-1} \end{bmatrix} \right) \\
&\qquad \left(\sqrt{\frac{\det \tilde{Y}}{\det Y}} \det(j(\gamma, Z)) \right)^{-k} |\tilde{a}|^k \\
&=^{(4.4)} \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} \phi \left(\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{b} & \\ & \tilde{b}^{-1} \end{bmatrix} \right) \left(\sqrt{\det \tilde{Y}} \det(j(\gamma, Z)) \right)^{-k} \\
&\qquad \left(\frac{\det(\tilde{Y})}{\tilde{y}} \right)^{k/2} \\
&= \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} \phi \left(\begin{bmatrix} 1 & \tilde{x} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\tilde{y}} & \\ & (\sqrt{\tilde{y}})^{-1} \end{bmatrix} \right) (\tilde{y})^{-k/2} \det(j(\gamma, Z))^{-k} \\
&= \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} f(\tilde{x} + i\tilde{y}) \det(j(\gamma, Z))^{-k}
\end{aligned}$$

$$= \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} f(\gamma\langle Z \rangle^*) \det(j(\gamma, Z))^{-k}.$$

Thus we obtained the desired Klingen Eisenstein series. $\square$

In the above proof, for each $p|N$, we have picked our ramified non-archimedean distinguished vectors which were supported only on $Q(\mathbb{Q}_p)1\Gamma_0^4(p^{2n_p})$. One can obtain different Klingen Eisenstein series by changing these local distinguished vectors.

Theorem 7.3.2. *Let S be a finite set of primes and $N = \prod_{p \in S} p^{n_p}$ be a positive integer. Let f be an elliptic cusp form of level N, weight k, i.e., $f \in S_k^1(\Gamma_0^2(N))$. We also assume f to be a newform. Let ϕ be the automorphic form associated with f and let (π, V_π) be the irreducible cuspidal automorphic representation of* $\mathrm{GL}(2, \mathbb{A})$ *generated by ϕ. Then there exists a global distinguished vector Φ in the global induced automorphic representation $|\cdot|^s \rtimes |\cdot|^{\frac{-s}{2}} \pi$ of* $\mathrm{GSp}(4, \mathbb{A})$*, for $s = k - 2$, such that,*

$$\bar{E}(Z) := \det(Y)^{-k/2} \sum_{\gamma \in Q(\mathbb{Q}) \backslash \mathrm{GSp}(4, \mathbb{Q})} (\Phi(\gamma b_Z))(1), \tag{7.14}$$

$$(\text{with } Z = X + iY \in \mathbb{H}_2, b_Z \text{ as in } (4.5)),$$

defines an Eisenstein series which is same as the Klingen Eisenstein series of level N with respect to $\Gamma_0^4(N)$, defined by

$$\begin{aligned} E(Z) &= \sum_{\gamma \in \left((s_1 s_2)^{-1} Q(\mathbb{Q}) s_1 s_2 \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} f(s_1 s_2 \gamma\langle Z \rangle^*) \det(j(s_1 s_2 \gamma, Z))^{-k} \\ &= \sum_{\gamma \in \left\{ \begin{bmatrix} & & & -1 \\ a & & b & \\ & 1 & & \\ c & & d & \end{bmatrix} \begin{bmatrix} -\mu & 1 & & \\ -k & l & 1 & \mu \\ l & & & 1 \\ -1 & & & \end{bmatrix} \cap \mathrm{Sp}(4, \mathbb{R}) \,|\, c, l, k \in N\mathbb{Z}, a, b, d, \mu \in \mathbb{Z} \right\} \backslash \Gamma_0^4(N)} f(s_1 s_2 \gamma\langle Z \rangle^*) \det(j(s_1 s_2 \gamma, Z))^{-k} \end{aligned} \tag{7.15}$$

Proof. We note that the last explicit description for $E(Z)$ follows from Lemma 7.2.3.

The rest of the proof is similar to Theorem 7.3.1, except that for each $p|N$ we pick our ramified non-archimedean distinguished vector $\tilde{\Phi}_p$ such that it is supported only on $Q(\mathbb{Q}_p)s_1s_2\mathrm{Si}(p^{n_p})$ and it is given by

$$\tilde{\Phi}_p\left(\begin{bmatrix} a & & b & * \\ * & t & * & * \\ c & & d & * \\ & & & t^{-1}(ad-bc) \end{bmatrix} s_1s_2\,\kappa\right) = |t^2(ad-bc)^{-1}|_p\,|t|_p^s\,|ad-bc|_p^{-\frac{s}{2}}\pi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\phi_p \tag{7.16}$$

where, $\kappa \in \mathrm{Si}(p^n), \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Q}_p)$ and $\phi_p \in \pi_p$ is such that

$$\pi_p(\begin{bmatrix} a & b \\ c & d \end{bmatrix})\phi_p = \phi_p \text{ for all } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^{n_p}\mathbb{Z}_p & \mathbb{Z}_p^\times \end{bmatrix}.$$

We omit further details. $\square$

7.3.1 Action of the Atkin-Lehner element

Let $N = \prod_{p\in S} p^{n_p}$ be a positive integer. Let Φ_p and $\tilde{\Phi}_p$ be the local non-archimedean distinguished vectors defined in (7.13) and (7.16). We recall that the support of Φ_p is $Q(\mathbb{Q}_p)1\mathrm{Si}(p^{n_p})$ and $\tilde{\Phi}_p$ is supported on the double coset $Q(\mathbb{Q}_p)s_1s_2\mathrm{Si}(p^{n_p})$. The Atkin-Lehner element $\omega_{n_p} = \begin{bmatrix} & & & -1 \\ & & 1 & \\ & p^{n_p} & & \\ -p^{n_p} & & & \end{bmatrix}$ acts on the subspace generated by Φ_p and $\tilde{\Phi}_p$. In fact, we note that $\omega_{n_p}\Phi_p$ is supported on $Q(\mathbb{Q}_p)s_1s_2\mathrm{Si}(p^{n_p})$. To see this we find all $g \in \mathrm{GSp}(4, \mathbb{Q}_p)$ such that $\omega_{n_p}\Phi_p(g) = \Phi_p(g\omega_{n_p}) \neq 0$. This is equivalent to finding all g such that $g\omega_{n_p} \in Q(\mathbb{Q}_p)1\mathrm{Si}(p^{n_p})$. Now using the identity

$$\omega_{n_p}^{-1} = \begin{bmatrix} & & -\frac{1}{p^{n_p}} & \\ & \frac{1}{p^{n_p}} & & \\ 1 & & & \\ & & & 1 \end{bmatrix} s_1s_2$$

and the fact that $\omega_{n_p}\mathrm{Si}(p^{n_p})\omega_{n_p}^{-1} \subseteq \mathrm{Si}(p^{n_p})$ we get that $g \in Q(\mathbb{Q}_p)s_1s_2\mathrm{Si}(p^{n_p})$.

We can now write $\omega_{n_p}\Phi_p = c\,\tilde{\Phi}_p$, for some constant $c \in \mathbb{C}$. To determine this constant c we evaluate both the sides on s_1s_2. Thus we get

$$c\,\tilde{\Phi}_p(s_1s_2) = \omega_{n_p}\Phi_p(s_1s_2) = \Phi_p(s_1s_2\omega_{n_p}) = \Phi_p(\begin{bmatrix} & & 1 & \\ & p^{n_p} & & \\ -p^{n_p} & & & \\ & & & 1 \end{bmatrix})$$
$$\implies c\,\phi_p = p^{-n_p(1+\frac{s}{2})}\pi_p\left(\begin{bmatrix} & 1 \\ -p^{n_p} & \end{bmatrix}\right)\phi_p = p^{-n_p(1+\frac{s}{2})}\pi_p\left(\begin{bmatrix} & 1 \\ p^{n_p} & \end{bmatrix}\right)\phi_p.$$

It follows from Theorem 3.2.2, [35], that the Atkin-Lehner eigenvalue of π_p could be given in terms of the local ε-factor attached to π_p. We obtain $c = \varepsilon(1/2, \pi_p)p^{-n_p(1+\frac{s}{2})}$. Similarly, we have $\omega_{n_p}\tilde{\Phi}_p = \varepsilon(1/2, \pi_p)p^{n_p(1+\frac{s}{2})}\,\Phi_p$. It is now clear that for the action of ω_{n_p} the two eigenvectors are $\Phi_p + c\,\tilde{\Phi}_p$ and $\Phi_p - c\,\tilde{\Phi}_p$ with with eigenvalues 1 and -1 respectively.

The above results could also be interpreted in terms of the action of the classical Atkin-Lehner operator u_p (as defined in [36]) on the classical Klingen Eisenstein series. We very briefly reproduce the definition of u_p here and refer readers to [36] for more details on the classical Atkin-Lehner action on the classical Siegel modular forms. Let p be a prime with $p|N$. We pick a matrix $\gamma_p \in \mathrm{Sp}(4, \mathbb{Z})$ such that $\gamma_p \equiv \left[\begin{smallmatrix} & J_1 \\ -J_1 & \end{smallmatrix}\right] \mod p^{n_p}$ with $J_1 = \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]$ and $\gamma_p \equiv \left[\begin{smallmatrix} I & \\ & I \end{smallmatrix}\right] \mod Np^{-n_p}$ with $I = \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right]$. Then the Atkin-Lehner element u_p is defined as

$$u_p := \gamma_p \begin{bmatrix} p^{n_p} & & & \\ & p^{n_p} & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

The Atkin-Lehner element u_p acts on a weight k Siegel modular form F with respect to $\Gamma_0^4(N)$ as $F \mapsto F|_k u_p$.

The following result is now evident from Theorem 7.3.1, Theorem 7.3.2 and the above discussion.

Proposition 7.3.3. *Let*

$$E(Z) = \sum_{\gamma \in \left(Q(\mathbb{Q}) \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} f(\gamma\langle Z\rangle^*) \det(j(\gamma, Z))^{-k}$$

and

$$\tilde{E}(Z) = \sum_{\gamma \in \left((s_1 s_2)^{-1} Q(\mathbb{Q}) s_1 s_2 \cap \Gamma_0^4(N)\right) \backslash \Gamma_0^4(N)} f(s_1 s_2 \gamma\langle Z\rangle^*) \det(j(s_1 s_2 \gamma, Z))^{-k}$$

be the classical Klingen Eisenstein series appearing in Theorem 7.3.1 and Theorem 7.3.2 respectively. Let ϵ *be the classical Atkin-Lehner eigenvalue of* f *and let* $c = \epsilon p^{-n_p(1+\frac{s}{2})}$ *with* $s = k - 2$*. Then the classical Atkin-Lehner operator* u_p *acts on the two dimensional space generated by* $E(Z)$ *and* $\tilde{E}(Z)$*. Moreover,* $E(Z) + c\tilde{E}(Z)$ *and* $E(Z) - c\tilde{E}(Z)$ *are eigenvectors for this action with eigenvalues* 1 *and* -1 *respectively.*

Chapter 8

Some double coset decompositions

8.1 Double coset decomposition $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p^n)$

In the following we determine a complete and minimal system of representatives for the double cosets $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p^n)$. The proof is algorithmic and essentially uses elementary number theory to establish the result. Before stating the main theorem, we state and prove several lemmas.

Notations: In this chapter we will use the symmetric version of GSp(4)**, instead of the classical version we have used so far**. Which means in this chapter we realize the group GSp(4) as

$$\mathrm{GSp}(4) := \{g \in \mathrm{GL}(4) \mid {}^t gJg = \lambda(g)J \text{ for some } \lambda(g) \in \mathrm{GL}(1)\},$$

with $J = \left[\begin{smallmatrix} & J_1 \\ -J_1 & \end{smallmatrix}\right]$ and $J_1 = \left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]$. We note that this version of GSp(4) is isomorphic to the classical version of GSp(4) which we have used in earlier chapters. We denote this isomorphism by the map $\jmath$ which interchanges the first two rows and the first two columns of any matrix.

We will also use the following notation:

$$g \sim h \iff Q(\mathbb{Q})\, g\, \Gamma_0(p^n) = Q(\mathbb{Q})\, h\, \Gamma_0(p^n), \qquad \text{for } g, h \in \mathrm{GSp}(4, \mathbb{Q}).$$

8.1.1 Some technical lemmas

Lemma 8.1.1. *Let p be a prime number and let x_1, x_2 be integers such that x_1, x_2 and p are pairwise co-prime. Further, assume y_1, y_2 to be integers such that y_1, y_2 and p are pairwise co-prime with $\gcd(x_1, y_1) = 1$. Let $x = x_1 x_2^{-1} p^{-r}$ and $y = y_1 y_2^{-1} p^{-s}$ with $r, s \geq 0$, $x_2 \neq 0$, $y_2 \neq 0$. Let $n \geq 1$. Let*

$$g = s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Then we have the following results.

1. *If $s > r$, then there exist integers η_1 and η_2 which are co-prime to p such that*

$$Q(\mathbb{Q}) g \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} & 1 & & \\ & & 1 & \\ & \eta_1 p^s & & 1 \\ \eta_2 p^{-r+2s} & & \eta_1 p^s & & 1 \end{bmatrix} \Gamma_0(p^n).$$

2. *If $s \leq r < n$, then there exists a non-zero integer x_3 co-prime to p such that,*

$$Q(\mathbb{Q}) g \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x_3 p^r & & & 1 \end{bmatrix} \Gamma_0(p^n),$$

and if $s \le r$ and $r \ge n$ then,

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n).$$

Proof. We have,

$$s_1s_2\begin{bmatrix}1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1\end{bmatrix} \sim \begin{bmatrix}-x & & -y & -1 \\ & & -1 & \\ & 1 & \frac{y^2}{x} & \frac{y}{x} \\ & & & -\frac{1}{x}\end{bmatrix} s_1s_2\begin{bmatrix}1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1\end{bmatrix} s_1$$
$$= \begin{bmatrix}1 & & & \\ & & -1 & \\ -\frac{y}{x} & 1 & & \\ \frac{1}{x} & & \frac{y}{x} & 1\end{bmatrix} \sim \begin{bmatrix}1 & & & \\ & 1 & & \\ -\frac{y}{x} & & 1 & \\ \frac{1}{x} & -\frac{y}{x} & & 1\end{bmatrix} s_2.$$

Now we prove the first part of the lemma.

Case 1: $\mathbf{s > r}$. Assume $s > r$. Let $\gcd(y_2, x_2) = \tau$. Let l_1 and l_2 be integers such that

$$l_1x_2y_1 + l_2p^{s-r}x_1y_2 = \tau.$$

Let

$$d_1 = \frac{l_1^2x_2y_2p^s}{\tau},\ d_2 = \frac{l_1l_2x_2y_2p^s}{\tau}.$$

It follows that

$$d_1x_2y_1 + d_2x_1y_2p^{s-r} = l_1p^sx_2y_2.$$

Then we have,

$$\begin{bmatrix}1 & & & \\ & & -1 & \\ -\frac{p^rx_2y_1}{p^sx_1y_2} & 1 & & \\ \frac{p^rx_2}{x_1} & & \frac{p^rx_2y_1}{p^sx_1y_2} & 1\end{bmatrix}$$

$$
= \begin{bmatrix} \frac{p^{-r+s}x_1y_2}{\tau} & -l_1 & & \\ & -\frac{l_1^2p^sx_2y_2}{\tau} & -\frac{p^{-r+s}x_1y_2}{\tau} & -l_1 \\ & \frac{p^{r-s}\tau}{x_1y_2} & & \\ & & & \frac{p^{r-s}\tau}{x_1y_2} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\frac{l_1p^sx_2y_2}{\tau} & & 1 & \\ \frac{p^{-r+2\,s}x_1x_2y_2^2}{\tau^2} & -\frac{l_1p^sx_2y_2}{\tau} & & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} l_2 & l_1 & & \\ -\frac{x_2y_1}{\tau} & \frac{p^{-r+s}x_1y_2}{\tau} & & \\ & & l_2 & -l_1 \\ & & \frac{x_2y_1}{\tau} & \frac{p^{-r+s}x_1y_2}{\tau} \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ -\frac{l_1p^sx_2y_2}{\tau} & & 1 & \\ \frac{p^{-r+2\,s}x_1x_2y_2^2}{\tau^2} & -\frac{l_1p^sx_2y_2}{\tau} & & 1 \end{bmatrix}.
$$

This completes the proof of the first part of the lemma with $\eta_1 = \frac{-l_1x_2y_2}{\tau}$ and $\eta_2 = \frac{x_1x_2y_2^2}{\tau^2}$.

Example 1: Assume:

```
p=31, r=2, s=3, n=4, x=5/13454 and y=143/2085370.
```

Then we have,

$$
s_1s_2 \begin{bmatrix} 1 & 2^{-1}\cdot 5^{-1}\cdot 7^{-1}\cdot 11\cdot 13\cdot 31^{-3} & & \\ & 1 & 2^{-1}\cdot 5\cdot 7^{-1}\cdot 31^{-2} & 2^{-1}\cdot 5^{-1}\cdot 7^{-1}\cdot 11\cdot 13\cdot 31^{-3} \\ & & 1 & \\ & & & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} -\frac{5}{13454} & & -\frac{143}{2085370} & -1 \\ & & -1 & \\ & 1 & \frac{20449}{1616161750} & \frac{143}{775} \\ & & & -\frac{13454}{5} \end{bmatrix} s_1s_2 \begin{bmatrix} 1 & & \frac{143}{2085370} & \\ & 1 & \frac{5}{13454} & \frac{143}{2085370} \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1
$$

$$
= \begin{bmatrix} 1 & & & \\ & -1 & & \\ -\frac{143}{775} & & 1 & \\ \frac{13454}{5} & \frac{143}{775} & & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & -\frac{168}{775} & & \\ & 1 & & \\ & & 1 & \frac{168}{775} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{775} & & & \\ & & 775 & \\ & -\frac{1}{775} & -58857482880 & \\ & & & 775 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{143}{775} & 1 & & \\ \frac{13454}{5} & & \frac{143}{775} & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 31 & -168 & & \\ -143 & 775 & & \\ 10860606960 & -58857482880 & 31 & 168 \\ 2085370 & & 143 & 775 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & & & \\ & 1 & & \\ 350342160 & & 1 & \\ 1616161750 & 350342160 & & 1 \end{bmatrix} \begin{bmatrix} 31 & -168 & & \\ -143 & 775 & & \\ & & 31 & 168 \\ & & 143 & 775 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ 2^4 \cdot 3 \cdot 5 \cdot 7^2 \cdot 31^3 & & 1 & \\ 2 \cdot 5^3 \cdot 7 \cdot 31^4 & 2^4 \cdot 3 \cdot 5 \cdot 7^2 \cdot 31^3 & & 1 \end{bmatrix}.
$$

Now we prove the second part of the lemma.

Case 2: $\mathbf{s \leq r}$. Assume $s \leq r$ and $\gcd(x_1, y_1) = 1$. Let $\gcd(y_2, x_2) = \tau$. Let l_1 and l_2 be integers such that

$$
l_1 x_2 y_1 p^{r-s} + l_2 x_1 y_2 = \tau. \tag{8.1}
$$

Let d_1 and d_2 be integers such that

$$
d_1 y_1 + d_2 x_1 p^s = -l_1 p^s. \tag{8.2}
$$

Let c_1 and c_2 be integers such that

$$
c_1 \frac{x_1 y_2}{\tau} + c_2 p^{\max(0, n-s)} = \frac{-d_1 y_2}{p^s}. \tag{8.3}
$$

Let

$$\beta = \frac{x_2}{\tau}(d_2\tau - c_1y_1). \tag{8.4}$$

It is easy to see that $\beta \in \mathbb{Z}$.

Further, if $r < n$ then we make the following choices. Let l_3 and l_4 be integers such that,

$$l_3\frac{x_2y_2}{\tau} + l_4p^{n-r} = \beta. \tag{8.5}$$

Let x_3 and x_4 be integers such that

$$x_3(l_3p^{r-s}\frac{x_2y_1}{\tau} - l_2) - x_4p^{n-r} = -\frac{x_2y_2}{\tau}. \tag{8.6}$$

Here the case when $r = s$ and $l_3\frac{x_2y_1}{\tau} - l_2$ is divisible by p needs a little explanation. Assume this is the case. Next if $p \nmid l_1$ then we pick integers α_1 and α_2 such that $\alpha_1x_1y_2 + \alpha_2p = l_1$. We note that for any integer α_1 we can replace l_1 and l_2 by $l_1 - \alpha_1x_1y_2$ and $l_2 + \alpha_1x_2y_1$ respectively. It is easy to see that it does not affect (8.1) and all the subsequent arguments remain valid. So we can assume that $p|l_1$ and $p \nmid l_2$ to begin with. Next we can also assume that $p^{s+1}|d_1$ and $p|d_2$ (see (8.2)). In picking the integer c_1 in (8.3) we can assume that $p|c_1$. Next it is clear that $p|\beta$ and hence $p|l_3$ (see (8.5)). But since $p \nmid l_2$ this shows that p does not divide $l_3\frac{x_2y_1}{\tau} - l_2$. Now x_3 and x_4 could be picked using the usual Euclidean algorithm such that (8.6) holds.

Next, If $r \geq n$ then let

$$l_3 = l_4 = x_3 = x_4 = 0. \tag{8.7}$$

Let us define

$$q_1 = \begin{bmatrix} \frac{\tau}{x_1y_2} & & \frac{l_3x_1y_2}{\tau} + l_1 & \\ & & \frac{x_1y_2}{\tau} & \\ & -\frac{\tau}{x_1y_2} & \frac{c_1p^sx_1y_2}{\tau} + d_1y_2 & -\frac{l_3x_1y_2}{\tau} - l_1 \\ & & & \frac{x_1y_2}{\tau} \end{bmatrix}$$

and

$$\gamma_1 = \left[\begin{matrix} & -\frac{l_3p^{r-s}x_2y_1}{\tau} - \frac{l_1p^{r-s}x_2y_1}{x_1y_2} + \frac{\tau}{x_1y_2} & \frac{l_3x_1y_2}{\tau} + l_1 \\ & -\frac{p^{r-s}x_2y_1}{\tau} & \frac{x_1y_2}{\tau} \\ -\frac{c_1p^rx_2y_1}{\tau} - \frac{d_1p^{r-s}x_2y_1}{x_1} & -\frac{l_3p^rx_2y_2}{\tau} - \frac{l_1p^rx_2}{x_1} & \frac{c_1p^sx_1y_2}{\tau} + d_1y_2 \\ \left(\frac{l_3p^{r-s}x_2y_1}{\tau} + \frac{l_1p^{r-s}x_2y_1}{x_1y_2} - \frac{\tau}{x_1y_2}\right)p^rx_3 & + \frac{p^rx_2y_2}{\tau} & -\left(\frac{l_3x_1y_2}{\tau} + l_1\right)p^rx_3 \end{matrix}\right.$$

$$\left.\begin{matrix} -\frac{l_3p^{r-s}x_2y_1}{\tau} - \frac{l_1p^{r-s}x_2y_1}{x_1y_2} + \frac{\tau}{x_1y_2} & -\frac{l_3x_1y_2}{\tau} - l_1 \\ \frac{p^{r-s}x_2y_1}{\tau} & \frac{x_1y_2}{\tau} \end{matrix}\right].$$

One can check that q_1 and γ_1 are indeed elements of $\mathrm{GSp}(4,\mathbb{Q})$. Next we verify that $\gamma_1 \in \Gamma_0(p^n)$. Let $c_{i,j}$ denote the (i,j) entry of γ_1. Then $c_{3,1} = -\frac{c_1p^rx_2y_1}{\tau} - \frac{d_1p^{r-s}x_2y_1}{x_1} - \frac{l_3p^rx_2y_2}{\tau} - \frac{l_1p^rx_2}{x_1}$. We will show that $p^n | c_{3,1}$. We have

$$\begin{aligned} c_{3,1} &= \frac{-p^rx_2}{x_1p^s}(l_1p^s + d_1y_1) - \frac{c_1p^rx_2y_1}{\tau} - \frac{l_3p^rx_2y_2}{\tau} \\ &= d_2p^rx_2 - \frac{c_1p^rx_2y_1}{\tau} - \frac{l_3p^rx_2y_2}{\tau} && \text{(follows from (8.2))} \\ &= \frac{p^rx_2}{\tau}\{(\tau d_2 - c_1y_1) - l_3y_2\} \\ &= \frac{p^rx_2}{\tau}(\frac{\tau\beta}{x_2} - l_3y_2) && \text{(follows from (8.4)).} \end{aligned}$$

If $r \geq n$ then $l_3 = 0$ and we are done. Otherwise if $r < n$ then we proceed as follows.

$$\begin{aligned} c_{3,1} &= \frac{p^rx_2}{\tau}(\frac{\tau\beta}{x_2} - l_3y_2) = \frac{p^r}{\tau}(\tau\beta - l_3x_2y_2) \\ &= \frac{p^r}{\tau}(\tau l_4p^{n-r}) && \text{(follows from (8.5))} \end{aligned}$$

$$= l_4 p^n.$$

Next, we consider $c_{4,1}$ with $r < n$. We have

$$\begin{aligned}
c_{4,1} &= \left(\frac{l_3 p^{r-s} x_2 y_1}{\tau} + \frac{l_1 p^{r-s} x_2 y_1}{x_1 y_2} - \frac{\tau}{x_1 y_2}\right) p^r x_3 + \frac{p^r x_2 y_2}{\tau} \\
&= \left(\frac{l_3 p^{r-s} x_2 y_1}{\tau} - l_2\right) p^r x_3 + \frac{p^r x_2 y_2}{\tau} \qquad \text{(follows from (8.1))} \\
&= x_4 p^n \qquad \text{(follows from (8.6))}.
\end{aligned}$$

Similarly, the remaining cases can be verified. One could also verify that

$$q_1 \begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{p^{r-s} x_2 y_1}{x_1 y_2} & 1 & & \\ \frac{p^r x_2}{x_1} & & \frac{p^{r-s} x_2 y_1}{x_1 y_2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^r x_3 & & & 1 \end{bmatrix} \gamma_1.$$

From this we get

$$\begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{p^{r-s} x_2 y_1}{x_1 y_2} & 1 & & \\ \frac{p^r x_2}{x_1} & & \frac{p^{r-s} x_2 y_1}{x_1 y_2} & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^r x_3 & & & 1 \end{bmatrix},$$

and the second part of the lemma follows.

Example 2: Assume:

`p=3, r=4, s=3, n=5, x=5/1134 and y=11/1890`.

Then we have,

$$
\begin{aligned}
& s_1 s_2 \begin{bmatrix} 1 & & 2^{-1}\cdot 3^{-3}\cdot 5^{-1}\cdot 7^{-1}\cdot 11 & \\ & 1 & 2^{-1}\cdot 3^{-4}\cdot 5\cdot 7^{-1} & 2^{-1}\cdot 3^{-3}\cdot 5^{-1}\cdot 7^{-1}\cdot 11 \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
\sim & \begin{bmatrix} -\frac{5}{1134} & & -\frac{11}{1890} & -1 \\ & & -1 & \\ & 1 & \frac{121}{15750} & \frac{33}{25} \\ & & & -\frac{1134}{5} \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 & & \frac{11}{1890} & \\ & 1 & \frac{5}{1134} & \frac{11}{1890} \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 \\
= & \begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{33}{25} & 1 & & \\ \frac{1134}{5} & & \frac{33}{25} & 1 \end{bmatrix} \\
\sim & \begin{bmatrix} 1 & -\frac{3}{25} & & \\ & 1 & & \\ & & 1 & \frac{3}{25} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{25} & & & \\ & & 25 & \\ & -\frac{1}{25} & -277830 & \\ & & & 25 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{33}{25} & 1 & & \\ \frac{1134}{5} & & \frac{33}{25} & 1 \end{bmatrix} \\
= & \begin{bmatrix} 4 & -3 & & \\ -33 & 25 & & \\ 367416 & -277830 & 4 & 3 \\ 5670 & & 33 & 25 \end{bmatrix} \\
\sim & \begin{bmatrix} 1 & -448224 & & \\ & 1 & & \\ & & 1 & 448224 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1111320 & 1 & \\ & & & 1 \end{bmatrix} \\
& \begin{bmatrix} 4 & -3 & & \\ -33 & 25 & & \\ 367416 & -277830 & 4 & 3 \\ 5670 & & 33 & 25 \end{bmatrix} \\
= & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 5670 & & & 1 \end{bmatrix} \begin{bmatrix} 14791396 & -11205603 & & \\ -33 & 25 & & \\ 2505123936 & 27505170 & 14791396 & 11205603 \\ -83867209650 & 63535769010 & 33 & 25 \end{bmatrix} \\
\sim & \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2\cdot 3^4\cdot 5\cdot 7 & & & 1 \end{bmatrix}.
\end{aligned}
$$

Example 3: Assume:

`p=3, r=6, s=3, n=5, x=5/10206 and y=11/1890.`

Then we have,

$$
s_1 s_2 \begin{bmatrix} 1 & & 2^{-1}\cdot 3^{-3}\cdot 5^{-1}\cdot 7^{-1}\cdot 11 & \\ & 1 & 2^{-1}\cdot 3^{-6}\cdot 5\cdot 7^{-1} & 2^{-1}\cdot 3^{-3}\cdot 5^{-1}\cdot 7^{-1}\cdot 11 \\ & & 1 & \\ & & & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} -\frac{5}{10206} & & -\frac{11}{1890} & -1 \\ & & -1 & \\ & 1 & \frac{121}{1750} & \frac{297}{25} \\ & & & -\frac{10206}{5} \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 & & \frac{11}{1890} & \\ & 1 & \frac{5}{10206} & \frac{11}{1890} \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1
$$

$$
= \begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{297}{25} & 1 & & \\ \frac{10206}{5} & & \frac{297}{25} & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & \frac{8}{25} & & \\ & 1 & & \\ & & 1 & -\frac{8}{25} \\ & & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{25} & & & \\ & & 25 & \\ & -\frac{1}{25} & 740880 & \\ & & & 25 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & -1 & \\ -\frac{297}{25} & 1 & & \\ \frac{10206}{5} & & \frac{297}{25} & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -95 & 8 & & \\ -297 & 25 & & \\ -8817984 & 740880 & -95 & -8 \\ 51030 & & 297 & 25 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -2963520 & 1 & \\ & & & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} -95 & 8 & & \\ -297 & 25 & & \\ -8817984 & 740880 & -95 & -8 \\ 51030 & & 297 & 25 \end{bmatrix}
$$

$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} -95 & 8 & & \\ -297 & 25 & & \\ 871347456 & -73347120 & -95 & -8 \\ 51030 & & 297 & 25 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

$\square$

Lemma 8.1.2. *Assume p to be a prime number. Let n be a positive integer. Let*

$$x = \frac{x_1}{p^r x_2},\ y = \frac{y_1}{p^s y_2} \text{ and } z = \frac{z_1}{p^t z_2},$$

where r, s, t are non negative integers, x_1, x_2, y_2, z_2 are non-zero integers and y_1, z_1 are integers. Let any two non-zero elements selected from the set $\{x_1, y_1, z_1, x_2, y_2, z_2, p\}$ be mutually co-prime except, possibly, when both the chosen elements belong to $\{x_2, y_2, z_2\}$. Let

$$g = s_1 s_2 s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Then there exist x', $y' \in \mathbb{Q}$ such that

$$Q(\mathbb{Q})\, g\, \Gamma_0(p^n) = Q(\mathbb{Q}) s_1 s_2 \begin{bmatrix} 1 & & y' & \\ & 1 & x' & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \Gamma_0(p^n).$$

Proof. We have

$$g \sim s_1 s_2 s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & & & \\ & 1 & & \\ -x & & 1 & \\ & -x & & 1 \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} x & & 1 & \\ & x & & 1 \\ & & \frac{1}{x} & \\ & & & \frac{1}{x} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ -x & & 1 & \\ & -x & & 1 \end{bmatrix} s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&= s_2 s_1 s_2 s_1 s_2 \begin{bmatrix} 1 & x^{-1} & & \\ & 1 & & \\ & & 1 & -x^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.
\end{aligned} \tag{8.8}
$$

Case 1: $\mathbf{y \neq 0, z \neq 0}$. Let us first consider the case when $y \neq 0$ and $z \neq 0$. Let

$$\alpha = \gcd(x_2 y_2, z_2).$$

Further,

$$
\text{if } s > t - r \quad \text{then let} \quad \begin{cases} -d_1 p^{n+t} + d_2 x_1 & = 1 \text{ with } d_1, d_2 \in \mathbb{Z}, \\ d_3 & = d_1 x_2 y_2 z_2 \, p^{n+s+r+t}, \\ c_1 & = (x_2 y_2 z_1 \, p^{r+s-t} + x_1 y_1 z_2) \alpha^{-1}, \\ \tau & = \gcd(d_3, c_1), \\ y_4 & = \alpha\tau \, x_1^{-1} z_2^{-1} \, p^{-s}, \\ -d_4 \tau p^n + d_5 x_1 & = 1 \text{ with } d_4, d_5 \in \mathbb{Z}, \\ d & = d_4 x_2 y_2 z_2 \alpha^{-1} p^{n+s+r}. \end{cases} \tag{8.9}
$$

Otherwise,

$$\text{if } s \le t-r \text{ then let } \begin{cases} -d_1p^{n+s+r} + d_2x_1 & = 1 \text{ with } d_1, d_2 \in \mathbb{Z}, \\ d_3 & = d_1x_2y_2z_2\, p^{n+s+r+t}, \\ c_1 & = (x_2y_2z_1 + x_1y_1z_2\, p^{-r-s+t})\alpha^{-1}, \\ \tau & = \gcd(d_3, c_1), \\ y_4 & = \alpha\tau\, x_1^{-1}z_2^{-1}\, p^{r-t}, \\ -d_4\tau p^n + d_5x_1 & = 1 \text{ with } d_4, d_5 \in \mathbb{Z}, \\ d & = d_4x_2y_2z_2\alpha^{-1}p^{n+t}. \end{cases} \tag{8.10}$$

Here we note that in order to define d_4 and d_5 as above in (8.9) and (8.10) we need $\gcd(\tau, x_1) = 1$. To see that $\gcd(\tau, x_1) = 1$, we first note $\gcd(d_1, x_1) = 1$ from the first statement in both (8.9) and (8.10). Then the next statement gives $\gcd(x_1, d_3) = 1$ on using the hypothesis of the lemma. Since $\tau = \gcd(d_3, c_1)$ we obtain $\gcd(x_1, \tau) = 1$.

Now let $c_2 = c_1\tau^{-1}$. Clearly $\gcd(d_3, c_2) = 1$. All the prime factors of d, except possibly those of d_4, are also the factors of d_3. This means that $\gcd(d, c_2) = \gcd(d_4, c_2)$. Let $\tau' = \gcd(d_4, c_2)$. Let τ'' be the largest factor of c_2 that is co-prime to τ'. If needed on replacing d_4 by $d_4 - \tau''x_1$ and d_5 by $d_5 - \tau''\tau p^n$, and noting that $\gcd(x_1, c_2) = 1$, we can assume that $\tau' = \gcd(d_4, c_2) = 1$. Then we pick integers a_1 and b such that

$$a_1d - bc_2 = 1. \tag{8.11}$$

Next, we set

$$a = a_1 + bz_1z_2^{-1}p^{-t},$$

$$
\begin{aligned}
c =& c_2 + dz_1 z_2^{-1} p^{-t},\\
x_4 =& \frac{cp^r x_2 y_4}{x_1} + \frac{p^r x_2 y_1}{p^s x_1} + \frac{dy_1 y_4}{p^s y_2}.
\end{aligned}
$$

Then

$$
g \sim s_2 s_1 s_2 s_1 s_2 \begin{bmatrix} 1 & x^{-1} & & \\ & 1 & & \\ & & 1 & -x^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{(from (8.8))}
$$

$$
= s_2 \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{p^r x_2}{x_1} \\ & -1 & -\frac{z_1}{p^t z_2} & -\frac{y_1}{p^s y_2} \\ -1 & -\frac{p^r x_2}{x_1} & -\frac{p^r x_2 z_1}{p^t x_1 z_2} - \frac{y_1}{p^s y_2} & -\frac{p^r x_2 y_1}{p^s x_1 y_2} \end{bmatrix} \qquad \text{(note that } s_2 \in Q(\mathbb{Q})\text{)}
$$

$$
\begin{aligned}
&\sim \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{p^r x_2}{x_1} \\ & -1 & -\frac{z_1}{p^t z_2} & -\frac{y_1}{p^s y_2} \\ -1 & -\frac{p^r x_2}{x_1} & -\frac{p^r x_2 z_1}{p^t x_1 z_2} - \frac{y_1}{p^s y_2} & -\frac{p^r x_2 y_1}{p^s x_1 y_2} \end{bmatrix}\\
&= \begin{bmatrix} & & & 1 \\ & -b & a - \frac{bz_1}{p^t z_2} & -\frac{ap^r x_2}{x_1} - \frac{by_1}{p^s y_2} \\ & -d & c - \frac{dz_1}{p^t z_2} & -\frac{cp^r x_2}{x_1} - \frac{dy_1}{p^s y_2} \\ -1 & -\frac{p^r x_2}{x_1} & -\frac{p^r x_2 z_1}{p^t x_1 z_2} - \frac{y_1}{p^s y_2} & -\frac{p^r x_2 y_1}{p^s x_1 y_2} \end{bmatrix}\\
&= \begin{bmatrix} & & 1 & \\ 1 & & \frac{y_4}{y_2} & \\ & & & 1 \\ & -1 & -x_4 & -\frac{y_4}{y_2} \end{bmatrix}\\
&\quad \begin{bmatrix} & -b & a - \frac{bz_1}{p^t z_2} & -\frac{ap^r x_2}{x_1} - \frac{by_1}{p^s y_2} - \frac{y_4}{y_2} \\ 1 & \frac{p^r x_2}{x_1} + \frac{dy_4}{y_2} & -\frac{cy_4}{y_2} + \frac{p^r x_2 z_1}{p^t x_1 z_2} + \frac{dy_4 z_1}{p^t y_2 z_2} + \frac{y_1}{p^s y_2} & \frac{cp^r x_2 y_4}{x_1 y_2} - x_4 + \frac{p^r x_2 y_1}{p^s x_1 y_2} + \frac{dy_1 y_4}{p^s y_2^2} \\ & & & 1 \\ & -d & c - \frac{dz_1}{p^t z_2} & -\frac{cp^r x_2}{x_1} - \frac{dy_1}{p^s y_2} \end{bmatrix}\\
&= \begin{bmatrix} & & 1 & \\ 1 & & \frac{y_4}{y_2} & \\ & & & 1 \\ & -1 & -\frac{x_4}{y_2} & -\frac{y_4}{y_2} \end{bmatrix} \begin{bmatrix} & -b & a_1 & -a_1 d_5 x_2 p^r \\ 1 & d_5 x_2 p^r & & \\ & & & 1 \\ & -d & c_2 & -c_2 d_5 x_2 p^r \end{bmatrix} && (8.12)\\
&\sim \begin{bmatrix} & & 1 & \\ 1 & & \frac{y_4}{y_2} & \\ & & & 1 \\ & -1 & -\frac{x_4}{y_2} & -\frac{y_4}{y_2} \end{bmatrix}
\end{aligned}
$$

$$= s_1 s_2 \begin{bmatrix} 1 & & y' & \\ & 1 & x' & y' \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

with

$$x' = \frac{x_4}{y_2} \text{ and } y' = \frac{y_4}{y_2}.$$

Here only the equality (8.12) needs an explanation. We assume that (8.10) holds. The details are similar for (8.9) and can easily be verified. Let $m_{i,j}$ denote the (i,j) entry of the second matrix appearing in (8.12). We begin with $m_{2,2}$. We have

$$\begin{aligned} p^r \frac{x_2}{x_1} + \frac{dy_4}{y_2} &= p^r \frac{x_2}{x_1} + \left(\frac{d_4 x_2 y_2 z_2 p^{n+t}}{\alpha y_2}\right)\left(\frac{\alpha\tau p^r}{x_1 z_2 p^t}\right) \\ &\qquad \text{(from the definition of } d \text{ and } y_4 \text{ in (8.10))} \\ &= \frac{x_2 p^r}{x_1}(1 + d_4 \tau p^n) \\ &= d_5 x_2 p^r \quad \text{(from the second last relation in (8.10))}, \end{aligned}$$

as desired. Next we consider $m_{4,4}$. We have in this case

$$\begin{aligned} \frac{cp^r x_2}{x_1} + \frac{dy_1}{y_2 p^s} &= \frac{c_2 p^r x_2}{x_1} + \frac{dx_2 z_1 p^r}{x_1 z_2 p^t} + \frac{dy_1}{p^s y_2} \qquad \text{(using the definition of c)} \\ &= \frac{c_2 p^r x_2}{x_1} + \frac{dp^{r-t}}{x_1 y_2 z_2}(x_2 y_2 z_1 + x_1 y_1 z_2 p^{t-r-s}) \\ &= \frac{c_2 p^r x_2}{x_1} + \frac{dp^{r-t}}{x_1 y_2 z_2}(\alpha c_1) \qquad \text{(using the definition of } c_1\text{)} \\ &= \frac{c_2 p^r x_2}{x_1} + \frac{dp^{r-t}}{x_1 y_2 z_2}(\alpha\tau c_2) \qquad \text{(using the definition of } c_2\text{)} \\ &= \frac{c_2 p^r x_2}{x_1} + \frac{c_2 d y_4}{y_2} \qquad \text{(using the definition of } y_4\text{)} \\ &= c_2\left(\frac{p^r x_2}{x_1} + \frac{dy_4}{y_2}\right) = c_2 m_{2,2} = c_2 d_5 x_2 p^r = -m_{4,4}, \end{aligned}$$

as we wanted. Next we consider $m_{1,4}$. We have

$$
\begin{aligned}
&\frac{ap^r x_2}{x_1} + \frac{by_1}{p^s y_2} + \frac{y_4}{y_2} \\
&= \frac{ap^r x_2}{x_1} + \frac{y_4}{y_2} + \frac{bp^{r-t}}{x_1 y_2 z_2}(\tau\alpha c_2 - x_2 y_2 z_1) \qquad \text{(using the definition of } c_2 \text{ and } c_1\text{)} \\
&= \frac{ap^r x_2}{x_1} + \frac{p^{r-t}\tau\alpha}{x_1 y_2 z_2} + \frac{bp^{r-t}}{x_1 y_2 z_2}(\tau\alpha c_2 - x_2 y_2 z_1) \qquad \text{(using the definition of } y_4\text{)} \\
&= \frac{ap^r x_2}{x_1} + \frac{-bx_2 z_1 p^{r-t}}{x_1 z_2} + \frac{\tau\alpha p^{r-t}}{x_1 y_2 z_2}(1 + bc_2) \\
&= \frac{ap^r x_2}{x_1} - \frac{bx_2 z_1 p^{r-t}}{x_1 z_2} + \frac{\tau\alpha p^{r-t}}{x_1 y_2 z_2}(a_1 d) \qquad \text{(from (8.11))} \\
&= \frac{ap^r x_2}{x_1} - \frac{bx_2 z_1 p^{r-t}}{x_1 z_2} + \frac{a_1 \tau x_2 d_4 p^{n+r}}{x_1} \qquad \text{(using the definition of } d\text{)} \\
&= \left(a - \frac{bz_1}{p^t z_2}\right)\frac{x_2 p^r}{x_1} + \frac{a_1 \tau x_2 d_4 p^{n+r}}{x_1} \\
&= \frac{a_1 x_2 p^r}{x_1} + \frac{a_1 \tau x_2 d_4 p^{n+r}}{x_1} \qquad \text{(using the definition of } a\text{)} \\
&= \frac{a_1 x_2 p^r}{x_1}(1 + d_4 \tau p^n) \\
&= a_1 d_5 x_2 p^r \qquad \text{(using the definition of } d_4 \text{ and } d_5\text{)} \\
&= -m_{1,4},
\end{aligned}
$$

as desired. Next we consider $m_{2,3}$ as follows.

$$
\begin{aligned}
&-\frac{cy_4}{y_2} + \frac{p^r x_2 z_1}{p^t x_1 z_2} + \frac{dy_4 z_1}{p^t y_2 z_2} + \frac{y_1}{p^s y_2} \\
&= \frac{p^r x_2 z_1}{p^t x_1 z_2} + \frac{y_1}{p^s y_2} + \frac{y_4}{y_2}\left(\frac{dz_1}{z_2 p^t} - c\right) \\
&= \frac{p^r x_2 z_1}{p^t x_1 z_2} + \frac{y_1}{p^s y_2} + \frac{y_4}{y_2}(-c_2) \qquad \text{(using the definition of } c\text{)} \\
&= 0 = m_{2,3} \qquad \text{(using the definition of } y_4 \text{ and } c_2\text{).}
\end{aligned}
$$

From the definition of x_4 it is easy to see that $m_{2,4} = 0$. One can easily check the remaining cases. This completes the proof in the case when both y and z are non-zero.

Case 2: $\mathbf{y} = \mathbf{0}$, $\mathbf{z} \neq \mathbf{0}$. If $y = 0$, $z \neq 0$ then we set $y_1 = 0$, $y_2 = 1$ and $s = 0$. It is easy to see that the previous proof remains valid for this case as well.

Case 3: $\mathbf{y} \neq \mathbf{0}$, $\mathbf{z} = \mathbf{0}$. If $z = 0$, $y \neq 0$ then we set $z_1 = 0$, $z_2 = 1$, and $t = 0$; and it is easy to see that the proof given in the first case remains valid for this case as well.

Case 4: $\mathbf{y} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$. Finally, we consider the case when both y and z are zero. In this case we make the following choices. Let

$$c = x_1, \quad d = -p^n$$

and select integers a and b such that

$$ad - bc = 1.$$

If $r \geq n$ then set

$$y_4 = \frac{x_2 p^{r-n}}{x_1}, \qquad x_4 = x_2 y_4 p^r, \qquad e = b x_2 p^{r-n}, \qquad f = 0,$$

otherwise if $r < n$ then set

$$y_4 = \frac{-a x_2 p^r}{x_1}, \qquad x_4 = x_2 y_4 p^r, \qquad e = 0, \qquad f = -b x_2 p^r.$$

Now we have

$$
\begin{aligned}
g &\sim s_2 s_1 s_2 s_1 s_2 \begin{bmatrix} 1 & x^{-1} & & \\ & 1 & & \\ & & 1 & -x^{-1} \\ & & & 1 \end{bmatrix} \qquad\qquad \text{(from (8.8))} \\
&\sim \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{p^r x_2}{x_1} \\ & -1 & & \\ -1 & -\frac{p^r x_2}{x_1} & & \end{bmatrix} \\
&= \begin{bmatrix} & & 1 & \\ 1 & & y_4 & \\ & & & 1 \\ & -1 & -x_4 & -y_4 \end{bmatrix} \begin{bmatrix} & -b & a & -\frac{ap^r x_2}{x_1} - y_4 \\ 1 & dy_4 + \frac{p^r x_2}{x_1} & -cy_4 & \frac{cp^r x_2 y_4}{x_1} - x_4 \\ & & & 1 \\ & -d & c & -\frac{cp^r x_2}{x_1} \end{bmatrix} \\
&= \begin{bmatrix} & & 1 & \\ 1 & & y_4 & \\ & & & 1 \\ & -1 & -x_4 & -y_4 \end{bmatrix} \begin{bmatrix} & -b & a & e \\ 1 & f & -p^{-n+r} x_2 & \\ & & & 1 \\ & p^n & x_1 & -p^r x_2 \end{bmatrix} \\
&\sim \begin{bmatrix} & & 1 & \\ 1 & & y_4 & \\ & & & 1 \\ & -1 & -x_4 & -y_4 \end{bmatrix} = s_1 s_2 \begin{bmatrix} 1 & & y' & \\ & 1 & x' & y' \\ & & 1 & \\ & & & 1 \end{bmatrix}
\end{aligned}
$$

with

$$x' = x_4 \text{ and } y' = y_4.$$

This completes the proof of the lemma.

Example 1: Assume:

```
p=13, r=2, s=3, t=7, n=8,

x=29/24167, y=3/101062 and z=7/1380467374.
```

Then we have,

$$s_1 s_2 s_1 s_2 \begin{bmatrix} 1 & \frac{24167}{29} & \frac{646}{19050187} & \frac{33}{1334} \\ & 1 & \frac{7}{1380467374} & \frac{3}{101062} \\ & & 1 & -\frac{24167}{29} \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{24167}{29} \\ & -1 & -\frac{7}{1380467374} & -\frac{3}{101062} \\ -1 & -\frac{24167}{29} & -\frac{646}{19050187} & -\frac{33}{1334} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & & & \\ & -\frac{8157958185984318059325}{1380467374} & -1165422597980799695023 & \\ & 22195217188012 & 4377110455206956994098 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} & & & 1 \\ & & 1 & -\frac{24167}{29} \\ & -1 & -\frac{7}{1380467374} & -\frac{3}{101062} \\ -1 & -\frac{24167}{29} & -\frac{646}{19050187} & -\frac{33}{1334} \end{bmatrix}$$

$$= \begin{bmatrix} & & 1 & \\ 1 & & \frac{2}{19050187} & \\ & & & 1 \\ & -1 & -\frac{270240673953745}{17342} & -\frac{2}{19050187} \end{bmatrix}$$

$$\begin{bmatrix} & 1165422597980799695023 & -86 & 39519979425764266 \\ 1 & 459534644485631 & & \\ & & & 1 \\ & -4377110455206956994098 & 323 & -148429690168858813 \end{bmatrix}$$

$$= s_1 s_2 \begin{bmatrix} 1 & & \frac{2}{19050187} & \\ & 1 & \frac{270240673953745}{17342} & \frac{2}{19050187} \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Example 2: Assume:

```
p=11, r=5, s=3, t=7, n=8,

x=23/23030293, y=3/42592 and z=7/370256249.
```

Then we have,

$$
\begin{aligned}
& s_1 s_2 s_1 s_2 \begin{bmatrix} 1 & \frac{23030293}{23} & \frac{353663}{18612704} & \frac{51909}{736} \\ & 1 & \frac{7}{370256249} & \frac{3}{42592} \\ & & 1 & -\frac{23030293}{23} \\ & & & 1 \end{bmatrix} \\
&= \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{23030293}{23} \\ & -1 & -\frac{7}{370256249} & -\frac{3}{42592} \\ -1 & -\frac{23030293}{23} & -\frac{353663}{18612704} & -\frac{51909}{736} \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & & & \\ & -\frac{150578094910926330850 93}{370256249} & -21511156449421455 35231 & \\ & -226589340960673 & -119851599397752538 97952 & \\ & & & 1 \end{bmatrix} \\
&\quad \begin{bmatrix} & & & 1 \\ & & 1 & -\frac{23030293}{23} \\ & -1 & -\frac{7}{370256249} & -\frac{3}{42592} \\ -1 & -\frac{23030293}{23} & -\frac{353663}{18612704} & -\frac{51909}{736} \end{bmatrix} \\
&= \begin{bmatrix} & & 1 & \\ 1 & & \frac{1}{18612704} & \\ & & & 1 \\ & -1 & \frac{171098394218543935}{13984} & -\frac{1}{18612704} \end{bmatrix} \\
&\quad \begin{bmatrix} & 21511156449421455 35231 & 63476 & 40873696328817023672 \\ 1 & -643923629857222 & & \\ & & & 1 \\ & 119851599397752538 97952 & 353663 & 22773196270619470 4186 \end{bmatrix} \\
&= s_1 s_2 \begin{bmatrix} 1 & & \frac{1}{18612704} & \\ & 1 & -\frac{171098394218543935}{13984} & \frac{1}{18612704} \\ & & 1 & \\ & & & 1 \end{bmatrix}.
\end{aligned}
$$

□

Lemma 8.1.3. *Assume n and s to be positive integers. Also let p be a prime number* *and η be an integer such that* $\gcd(\eta, p) = 1$. *Then*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \eta p^s & & 1 & \\ & \eta p^s & & 1 \end{bmatrix} \Gamma_0(p^n).$$

Proof. If $s \geq n$ then, of course, both sides equal $Q(\mathbb{Q})1\Gamma_0(p^n)$. Therefore, in the following we assume that $s < n$. Next we consider the case when $n \geq 2s$ and make the following choices. Since $\gcd(\eta, p) = 1$, there exist integers α_1 and β_1 such that $\alpha_1\eta + \beta_1 p^{n-s} = 1$. Further, we also have $\gcd(\alpha_1, p^{n-s}) = 1$, so there exist β_2' and β_3' such that $\alpha_1\beta_2' + \beta_3' p^{n-s} = 1$. Let $\beta_2 = \beta_1\beta_2'$ and $\beta_3 = -\beta_1\beta_3'$. Now set

$$a = \frac{1 - \beta_1 p^{n-s}}{\eta} = \alpha_1,\ b = p^{n-2s}\beta_3,\ c = p^n,\ d = \eta + \beta_2 p^{n-s}.$$

We also check that

$$\begin{aligned} ad - bc &= \alpha_1(\eta + \beta_2 p^{n-s}) - \beta_3 p^{2n-2s} = 1 - \beta_1 p^{n-s} + \alpha_1\beta_2 p^{n-s} - \beta_3 p^{2n-2s} \\ &= 1 - \beta_1 p^{n-s}(1 - \alpha_1\beta_2' - \beta_3' p^{n-s}) = 1. \end{aligned}$$

On the other hand if $2s > n$, then we make the following choices. Since $\gcd(\eta, p) = 1$ there exist integers α_1 and β_1 such that $\alpha_1\eta + \beta_1 p^{n-s} = 1$. Further, we also have $\gcd(\alpha_1, p^s) = 1$, so there exist β_2' and β_3' such that $\alpha_1\beta_2' + \beta_3' p^s = 1$. Let $\beta_2 = \beta_1\beta_2'$ and $\beta_3 = -\beta_1\beta_3'$. Now set

$$a = \frac{1 - \beta_1 p^{n-s}}{\eta} = \alpha_1,\ b = \beta_3,\ c = p^n\ d = \eta + \beta_2 p^{n-s}.$$

Next we note,

$$ad - bc = \alpha_1(\eta + \beta_2 p^{n-s}) - \beta_3 p^n = 1 - \beta_1 p^{n-s} + \alpha_1\beta_2 p^{n-s} - \beta_3 p^n$$

$$= 1 - \beta_1 p^{n-s} + \alpha_1\beta_1\beta_2' p^{n-s} + \beta_1\beta_3' p^n = 1 - \beta_1 p^{n-s}(1 - \alpha_1\beta_2' - \beta_3' p^s) = 1.$$

Now the lemma follows from the following calculations.

$$\begin{aligned}
Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix} \Gamma_0(p^n) &= Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix} \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \eta p^s & & 1 & \\ & \eta p^s & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ bp^s & a & b & \\ dp^s - \eta p^s & c & d & \\ -b\eta (p^s)^2 & -a\eta p^s + p^s & -b\eta p^s & 1 \end{bmatrix} \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \eta p^s & & 1 & \\ & \eta p^s & & 1 \end{bmatrix} \Gamma_0(p^n).
\end{aligned}$$

□

Lemma 8.1.4. *Assume s, r and n to be positive integers with $0 < s \leq n$. Also let p be a prime number and η_1, η_2 be integers such that* $\gcd(\eta_1, p) = 1$, $\gcd(\eta_2, p) = 1$. *Then*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & p^s & 1 & \\ \eta_2 p^r & p^s & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \eta_1 p^s & & 1 & \\ \eta_2 p^r & \eta_1 p^s & & 1 \end{bmatrix} \Gamma_0(p^n).$$

Proof. Let

$$z_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & p^s & 1 & \\ \eta_2 p^r & p^s & & 1 \end{bmatrix} \text{ and } z_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ \eta_1 p^s & & 1 & \\ \eta_2 p^r & \eta_1 p^s & & 1 \end{bmatrix}.$$

Then we note that,

$$z_2^{-1} \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix} z_1 = \begin{bmatrix} 1 & & & \\ bp^s & a & b & \\ dp^s - p^s\eta_1 & c & d & \\ -b(p^s)^2\eta_1 & -ap^s\eta_1 + p^s & -bp^s\eta_1 & 1 \end{bmatrix}.$$

Now, the result follows by proceeding as in the proof of Lemma 8.1.3. □

Lemma 8.1.5. *Assume n to be a positive integer and s to be a non-negative integer.*

Also let p be a prime number. Let $y_1,\ y_2 \in \mathbb{Z}$ such that $p,\ y_1,\ y_2$ are pairwise co-prime. Then we have the following results.

1. *If $s < n$ then there exists an integer b_1 with $\gcd(b_1, p) = 1$, such that*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_1}{y_2}p^s & & 1 & \\ & \frac{y_1}{y_2}p^s & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ b_1p^s & & 1 & \\ & b_1p^s & & 1 \end{bmatrix} \Gamma_0(p^n).$$

2. *If $s \geq n$ then*

$$Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_1}{y_2}p^s & & 1 & \\ & \frac{y_1}{y_2}p^s & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n).$$

Proof. Let α_1 and β_1 be integers such that

$$\alpha_1 p^s y_1 + \beta_1 y_2 = 1.$$

If $0 < s < n$ then set:

$$\begin{aligned} \alpha = \alpha_1,\ \beta &= \beta_1, \\ b_1,\ b_2 \in \mathbb{Z}, \text{ such that } \ b_1\beta + b_2p^{n-s} &= y_1, \\ b &= b_1p^s. \end{aligned}$$

Otherwise, if $s \geq n$ then set:

$$\begin{aligned} \alpha = \alpha_1,\ \beta = \beta_1, \\ b_2 = y_1p^{s-n},\ b_1 = 0,\ b = 0. \end{aligned}$$

If $s = 0$ then set:

$$\alpha = \begin{cases} \alpha_1 - y_2 & \text{if } \ p \mid \beta_1 \\ \alpha_1 & \text{if } \ p \nmid \beta_1 \end{cases},$$

$$\beta = \begin{cases} \beta_1 + y_1 & \text{if } \ p \mid \beta_1 \\ \beta_1 & \text{if } \ p \nmid \beta_1 \end{cases},$$

$$b_1, b_2 \in \mathbb{Z}, \text{ such that } \ b_1\beta + b_2 p^n = y_1,$$
$$b = b_1.$$

We note that, for each of the cases considered above, i.e., whenever $s \geq 0$, the following holds,

$$-b\beta + p^s y_1 = (-b_1\beta + y_1)p^s = b_2 p^n.$$

Then,

$$\begin{aligned}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_1}{y_2}p^s & & 1 & \\ & \frac{y_1}{y_2}p^s & & 1 \end{bmatrix} &\sim \begin{bmatrix} y_2^{-1} & & -\alpha & \\ & y_2^{-1} & & -\alpha \\ & & y_2 & \\ & & & y_2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_1}{y_2}p^s & & 1 & \\ & \frac{y_1}{y_2}p^s & & 1 \end{bmatrix} = \begin{bmatrix} \beta & & -\alpha & \\ & \beta & & -\alpha \\ p^s y_1 & & y_2 & \\ & p^s y_1 & & y_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ & 1 & & \\ b & & 1 & \\ & b & & 1 \end{bmatrix} \begin{bmatrix} \beta & & -\alpha & \\ & \beta & & -\alpha \\ -b\beta + p^s y_1 & & \alpha b + y_2 & \\ & -b\beta + p^s y_1 & & \alpha b + y_2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ b_1 p^s & & 1 & \\ & b_1 p^s & & 1 \end{bmatrix}.
\end{aligned}$$

This completes the proof of the lemma.

$\square$

Lemma 8.1.6. *Assume n to be a positive integer, r to be a non-negative integer and p to be a prime number. Let $x_1,\, x_2 \in \mathbb{Z}$ such that $p,\, x_1,\, x_2$ are pairwise co-prime. Then we have the following results.*

1. *If $r < n$ then there exists an integer c_1 with $\gcd(c_1, p) = 1$, such that*

$$Q(\mathbb{Q}) \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ \frac{x_1}{x_2}p^r & & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ c_1 p^r & & & 1 \end{bmatrix} \Gamma_0(p^n).$$

2. *If $r \geq n$ then*

$$Q(\mathbb{Q}) \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ \frac{x_1}{x_2}p^r & & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) 1 \Gamma_0(p^n).$$

Proof. Let α_1 and β_1 be integers such that

$$\alpha_1 p^r x_1 + \beta_1 x_2 = 1.$$

If $0 < r < n$ then set:

$$\begin{aligned} \alpha = \alpha_1,\, \beta &= \beta_1, \\ c_1,\, c_2 \in \mathbb{Z}, \text{ such that } \; c_1\beta + c_2 p^{n-r} &= x_1, \\ c &= c_1 p^r. \end{aligned}$$

Otherwise, if $r \geq n$ then set:

$$\alpha = \alpha_1,\, \beta = \beta_1,$$

$$c_2 = x_1 p^{r-n},\, c_1 = 0,\, c = 0.$$

If $r = 0$ then set:

$$\alpha = \begin{cases} \alpha_1 - x_2 & \text{if } p \mid \beta_1 \\ \alpha_1 & \text{if } p \nmid \beta_1 \end{cases},$$

$$\beta = \begin{cases} \beta_1 + x_1 & \text{if } p \mid \beta_1 \\ \beta_1 & \text{if } p \nmid \beta_1 \end{cases},$$

$$c_1,\, c_2 \in \mathbb{Z}, \text{ such that } c_1\beta + c_2 p^n = x_1,$$

$$c = c_1.$$

We note that, for $r \geq 0$ the following holds,

$$-c\beta + p^r x_1 = (-c_1\beta + x_1)p^r = c_2 p^n.$$

Then,

$$\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ \frac{x_1}{x_2}p^r & & & 1 \end{bmatrix} \sim \begin{bmatrix} x_2^{-1} & & & -\alpha \\ & 1 & & \\ & & 1 & \\ & & & x_2 \end{bmatrix} \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ \frac{x_1}{x_2}p^r & & & 1 \end{bmatrix} = \begin{bmatrix} & \beta & & -\alpha \\ & & 1 & \\ & & & 1 \\ p^r x_1 & & & x_2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ c & & & 1 \end{bmatrix} \begin{bmatrix} & \beta & & -\alpha \\ & & 1 & \\ & & & 1 \\ -\beta c + p^r x_1 & & & \alpha c + x_2 \end{bmatrix} \sim \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ c_1 p^r & & & 1 \end{bmatrix}.$$

This completes the proof of the lemma.

∎

Lemma 8.1.7. *Assume n to be a positive integer and p to be a prime number. Let x, y be non-zero integers co-prime to p. Then we have the following results.*

1.

$$Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\&&1&\\x&&&1\end{bmatrix}\Gamma_0(p^n)=Q(\mathbb{Q})s_1s_2\Gamma_0(p^n).$$

2.

$$Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\y&&1&\\&y&&1\end{bmatrix}\Gamma_0(p^n)=Q(\mathbb{Q})s_1s_2\Gamma_0(p^n).$$

3.

$$Q(\mathbb{Q})\begin{bmatrix}1&&&\\&1&&\\y&&1&\\x&y&&1\end{bmatrix}\Gamma_0(p^n)=Q(\mathbb{Q})s_1s_2\Gamma_0(p^n).$$

Proof. Let k_1 and k_2 be integers such that

$$k_1x+k_2p^n=1.$$

Then we have,

$$\begin{bmatrix}1&&&\\&1&&\\&&1&\\x&&&1\end{bmatrix}\sim\begin{bmatrix}1&&&\\&1&&\\&&1&\\x&&&1\end{bmatrix}\begin{bmatrix}&&-k_1&1\\1&&&\\&&&1\\&k_1x-1&-x&\end{bmatrix}$$
$$=\begin{bmatrix}1&&&k_1\\&1&&\\&&1&\\&&&1\end{bmatrix}s_1s_2\sim s_1s_2.$$

This completes the proof of the first part of lemma.

Now, let l_1 and l_2 be integers such that $l_1y+l_2p^n=-1$. Then,

$$\begin{bmatrix}1&&&\\&1&&\\y&&1&\\&y&&1\end{bmatrix}\sim\begin{bmatrix}1&&&\\&1&&\\y&&1&\\&y&&1\end{bmatrix}\begin{bmatrix}&l_1&&1\\&&l_1&&1\\-l_1y-1&&-y&\\&-l_1y-1&&-y\end{bmatrix}$$

$$=s_2\begin{bmatrix}1 & l_1 & & \\ & 1 & & \\ & & 1 & -l_1 \\ & & & 1\end{bmatrix} s_1s_2 \sim s_1s_2.$$

This completes the proof of the second part of lemma. Finally we have,

$$\begin{bmatrix}1 & & & \\ & 1 & & \\ y & & 1 & \\ x & y & & 1\end{bmatrix} \sim \begin{bmatrix}1 & & & \\ & 1 & & \\ y & & 1 & \\ x & y & & 1\end{bmatrix}\begin{bmatrix} l_1 & & 1 & \\ l_1^2x & l_1 & & 1 \\ -l_1y-1 & & -y & \\ -l_1^2xy-l_1x & -l_1y-1 & -x & -y\end{bmatrix}$$
$$=\begin{bmatrix}1 & & & \\ & l_1^2x & 1 & \\ & -1 & & \\ & & & 1\end{bmatrix}\begin{bmatrix}1 & l_1 & & \\ & 1 & & \\ & & 1 & -l_1 \\ & & & 1\end{bmatrix} s_1s_2 \sim s_1s_2.$$

This completes the proof of the last part of lemma. $\square$

Lemma 8.1.8. *Assume n and r to be integers such that $0 < r < n$. Let p be a prime number and $x,\ y \in \mathbb{Z}$ such that $\gcd(x,p) = \gcd(y,p) = 1$. Let*

$$g_1(x,p,r) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^rx & & & 1\end{bmatrix} \text{ and } g_1(y,p,r) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^ry & & & 1\end{bmatrix}.$$

Then,

$$Q(\mathbb{Q})g_1(x,p,r)\Gamma_0(p^n) = Q(\mathbb{Q})g_1(y,p,r)\Gamma_0(p^n),$$

if and only if

$$x \equiv y \mod p^f$$

where $f = \min(r, n-r)$.

Proof. It is clear that $g_1(x,p,r)\Gamma_0(p^n) \sim g_1(y,p,r)\Gamma_0(p^n)$ if and only if there exists an element

$$q = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \frac{ad-bc}{t}\end{bmatrix}\begin{bmatrix} 1 & l & \mu & k \\ & 1 & & \mu \\ & & 1 & -l \\ & & & 1\end{bmatrix} \in Q(\mathbb{Q}),$$

such that $g_1(y,p,r)^{-1}qg_1(x,p,r) \in \Gamma_0(p^n)$. Suppose $g_1(y,p,r)^{-1}qg_1(x,p,r) \in \Gamma_0(p^n)$.

Then on comparing the multiplier on both the sides we get $ad - bc = 1$. Now we have,

$$g_1(y,p,r)^{-1} q g_1(x,p,r) = \begin{bmatrix} kp^r tx + t & lt & \mu t & kt \\ -(bl - a\mu)p^r x & a & b & -bl + a\mu \\ -(dl - c\mu)p^r x & c & d & -dl + c\mu \\ -\left(kp^r ty - \frac{1}{t}\right)p^r x - p^r ty & -lp^r ty & -\mu p^r ty & -kp^r ty + \frac{1}{t} \end{bmatrix}.$$

We conclude $t = \pm 1$. We also need the condition that

$$\begin{aligned} -\left(kp^r ty - \frac{1}{t}\right)p^r x - p^r ty &\equiv 0 \mod p^n \\ \implies t^2 y - x \equiv 0 \mod p^f \implies y - x &\equiv 0 \mod p^f. \end{aligned}$$

Conversely, we show that if $y - x \equiv 0 \mod p^f$ then $g_1(x,p,r)$ and $g_1(y,p,r)$ lie in the same double coset. Suppose $x - y = k_2 p^f$. As $\gcd(p^{n-r-f}, xyp^{r-f}) = 1$ there exist integers k and k_1 such that $kxyp^{r-f} + k_1 p^{n-r-f} = k_2$. So we obtain,

$$-(kp^r y - 1)p^r x - p^r y = k_1 p^n.$$

Therefore,

$$g_1(y,p,r)^{-1} \begin{bmatrix} 1 & & & k \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} g_1(x,p,r) = $$
$$\begin{bmatrix} kp^r x + 1 & & & k \\ & 1 & & \\ & & 1 & \\ -(kp^r y - 1)p^r x - p^r y & & & -kp^r y + 1 \end{bmatrix} \in \Gamma_0(p^n).$$

This means that $g_1(x,p,r)$ and $g_1(y,p,r)$ lie in the same double coset. This completes the proof of the lemma. □

Lemma 8.1.9. *Assume s, r and n to be integers such that $n \geq 1$, $0 < s < n$. Let*

p be a prime number and $x,\ y \in \mathbb{Z}$ such that $\gcd(x,p) = \gcd(y,p) = 1$. *Let*

$$g_3(p,x,r,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ xp^r & p^s & & 1 \end{bmatrix},\ g_2(p,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix}$$

$$\textit{and}\ \ g_3(p,y,r,s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ yp^r & p^s & & 1 \end{bmatrix}.$$

1. *If $r < n$ and $0 < s < r < 2s$ and $f = \min(2s-r, n-r)$ then*

$$Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) = Q(\mathbb{Q})g_3(p,y,r,s)\Gamma_0(p^n) \iff x \equiv y \mod p^f.$$

2. *If $2s \leq r$ then*

$$Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) = Q(\mathbb{Q})g_2(p,s)\Gamma_0(p^n).$$

3. *If $r \geq n$ then*

$$Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) = Q(\mathbb{Q})g_2(p,s)\Gamma_0(p^n).$$

Proof. It is clear that $Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) = Q(\mathbb{Q})g_3(p,y,r,s)\Gamma_0(p^n)$ if and only if there exists an element

$$q = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & \frac{ad-bc}{t} \end{bmatrix} \begin{bmatrix} 1 & l & \mu & k \\ & 1 & & \mu \\ & & 1 & -l \\ & & & 1 \end{bmatrix} \in Q(\mathbb{Q}),$$

such that $g_3(p,y,r,s)^{-1}qg_3(p,x,r,s) \in \Gamma_0(p^n)$. Suppose $g_3(y,p,r,s)^{-1}qg_3(x,p,r,s) \in \Gamma_0(p^n)$. Then on comparing the multiplier of the matrices on both the sides, we see that $ad - bc = 1$. Then on writing the matrix on the left explicitly it also follows that $t = \pm 1$. We can assume $t = 1$. Now,

$$g_3(p,y,r,s)^{-1} q g_3(p,x,r,s) =$$
$$\begin{bmatrix} kp^r x + \mu p^s + 1 & kp^s + l & \mu & k \\ -(bl - a\mu)p^r x + bp^s & -(bl-a\mu)p^s + a & b & -bl + a\mu \\ -(dl - c\mu + kp^s)p^r x - (\mu p^s - d)p^s - p^s & -(dl - c\mu + kp^s)p^s - lp^s + c & -\mu p^s + d & -dl + c\mu - kp^s \\ (blp^s - a\mu p^s - kp^r y + 1)p^r x - (\mu p^r y + bp^s)p^s - p^r y & -lp^r y + (blp^s - a\mu p^s - kp^r y + 1)p^s - ap^s & -\mu p^r y - bp^s & blp^s - a\mu p^s - kp^r y + 1 \end{bmatrix}.$$

Then on looking at the lowest left entry we get,

$$\begin{aligned} (blp^s - a\mu p^s - kp^r y + 1)p^r x - (\mu p^r y + bp^s)p^s - p^r y &\equiv 0 \mod p^n \\ \implies p^r(x-y) + (bl - a\mu)xp^{r+s} - kxyp^{2r} - \mu y p^{r+s} - bp^{2s} &\equiv 0 \mod p^n \\ \implies x - y + (bl - a\mu)xp^s - kxyp^r - \mu y p^s - bp^{2s-r} &\equiv 0 \mod p^{n-r} \\ \implies x - y &\equiv 0 \mod p^f. \end{aligned}$$

Conversely, we show that if $y - x \equiv 0 \mod p^f$ then $g_3(p,x,r,s)$ and $g_3(p,y,r,s)$ lie in the same double-coset. If $f = n - r$ then, let $x - y = k_1 p^{n-r}$.

$$\begin{aligned} &Q(\mathbb{Q})g_3(p,x,r,s)\Gamma_0(p^n) \\ &= Q(\mathbb{Q})g_3(p,y,r,s)\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^r x - p^r y & & & 1 \end{bmatrix}\Gamma_0(p^n) \\ &= Q(\mathbb{Q})g_3(p,y,r,s)\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^n k_1 & & & 1 \end{bmatrix}\Gamma_0(p^n) \\ &= Q(\mathbb{Q})g_3(p,y,r,s)\Gamma_0(p^n). \end{aligned}$$

On the other hand if $f = 2s - r$ or equivalently $2s \leq n$, then let $x - y = k_2 p^{2s-r}$.

$$\begin{aligned}
&Q(\mathbb{Q}) g_3(p, x, r, s) \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & k_2 & \\ & & 1 & \\ & & & 1 \end{bmatrix} g_3(p, x, r, s) \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) g_3(p, y, r, s) \begin{bmatrix} 1 & & & \\ k_2 p^s & 1 & & k_2 \\ & & 1 & \\ & & -k_2 p^s & 1 \end{bmatrix} \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) g_3(p, y, r, s) \Gamma_0(p^n).
\end{aligned}$$

This means that $g_3(p, x, r, s)$ and $g_3(p, y, r, s)$ lie in the same double coset and the first part of lemma follows. Next,

$$\begin{aligned}
&Q(\mathbb{Q}) g_3(p, x, r, s) \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & p^{r-2\,s} x & \\ & & 1 & \\ & & & 1 \end{bmatrix} g_3(p, x, r, s) \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) g_2(p, s) \begin{bmatrix} 1 & & & \\ p^{r-2\,s} p^s x & 1 & & p^{r-2\,s} x \\ & & 1 & \\ & & -p^{r-2\,s} p^s x & 1 \end{bmatrix} \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) g_2(p, s) \Gamma_0(p^n)
\end{aligned}$$

This completes the proof of the second part of lemma. Finally, the last part of the lemma follows from the calculation

$$g_2(p, s)^{-1} g_3(p, x, r, s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^r x & & & 1 \end{bmatrix} \in \Gamma_0(p^n).$$

□

Theorem 8.1.10. ***(Double coset decomposition)***. *Assume $n \geq 1$. A complete and minimal system of representatives for the double cosets $Q(\mathbb{Q})\backslash \mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p^n)$ is given by*

$$1, \qquad s_1 s_2, \qquad g_1(p,\gamma,r) = \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ \gamma p^r &&& 1 \end{bmatrix}, \ 1 \leq r \leq n-1,$$

$$g_2(p,s) = \begin{bmatrix} 1 &&& \\ & 1 && \\ p^s && 1 & \\ & p^s && 1 \end{bmatrix}, \ 1 \leq s \leq n-1,$$

$$g_3(p,\delta,r,s) = \begin{bmatrix} 1 &&& \\ & 1 && \\ p^s && 1 & \\ \delta p^r & p^s && 1 \end{bmatrix}, \ 1 \leq s, r \leq n-1, \ s < r < 2s,$$

where γ, δ runs through elements in $(\mathbb{Z}/f_1\mathbb{Z})^\times$ and $(\mathbb{Z}/f_2\mathbb{Z})^\times$ respectively with $f_1 = \min(r, n-r)$ and $f_2 = \min(2s-r, n-r)$. The total number of representatives given above is

$$C_1(p^n) = \begin{cases} \dfrac{p^{\frac{n}{2}+1} + p^{\frac{n}{2}} - 2}{p-1} & \text{if } n \text{ is even,} \\[2ex] \dfrac{2(p^{\frac{n+1}{2}} - 1)}{p-1} & \text{if } n \text{ is odd.} \end{cases} \tag{8.13}$$

Proof. First we prove completeness. We begin by writing

$$\begin{aligned} \mathrm{GSp}(4,\mathbb{Q}) = Q(\mathbb{Q}) \sqcup Q(\mathbb{Q}) s_1 & \begin{bmatrix} 1 & * && \\ & 1 && \\ && 1 & * \\ &&& 1 \end{bmatrix} \\ \sqcup Q(\mathbb{Q}) s_1 s_2 \begin{bmatrix} 1 && * & \\ & 1 & * & * \\ && 1 & \\ &&& 1 \end{bmatrix} & \sqcup Q(\mathbb{Q}) s_1 s_2 s_1 \begin{bmatrix} 1 & * & * & * \\ & 1 && * \\ && 1 & * \\ &&& 1 \end{bmatrix}, \end{aligned} \tag{8.14}$$

by using the Bruhat decomposition. We consider all the different possibilities.

First Cell: If $g \in Q(\mathbb{Q})$, then, of course, $Q(\mathbb{Q})g\Gamma_0(p^n)$ is represented by 1.

Second Cell: Assume that g is in the second cell. Then we may assume that

$$g = s_1 \begin{bmatrix} 1 & \frac{x_1}{x_2} & & \\ & 1 & & \\ & & 1 & -\frac{x_1}{x_2} \\ & & & 1 \end{bmatrix}, \qquad x_1,\, x_2 \in \mathbb{Z}, \text{ and } \gcd(x_1, x_2) = 1.$$

As $\gcd(x_1, x_2) = 1$, there exist integers l_1 and l_2 such that $-l_1x_1 + l_2x_2 = 1$.

$$\begin{aligned} Q(\mathbb{Q})g\Gamma_0(p^n) &= Q(\mathbb{Q}) \begin{bmatrix} \frac{1}{x_2} & l_1 & & \\ & x_2 & & \\ & & \frac{1}{x_2} & -l_1 \\ & & & x_2 \end{bmatrix} g\Gamma_0(p^n) \\ &= Q(\mathbb{Q}) \begin{bmatrix} l_1 & \frac{l_1x_1}{x_2} + \frac{1}{x_2} & & \\ x_2 & x_1 & & \\ & & -l_1 & \frac{l_1x_1}{x_2} + \frac{1}{x_2} \\ & & x_2 & -x_1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n). \end{aligned}$$

Third Cell: Next let g be an element in the third cell. We may assume that

$$g = s_1s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}, \qquad x, y \in Q(\mathbb{Q}).$$

The following calculation shows that we can replace x, y by $x+1$ and $y+1$ respectively.

$$\begin{aligned} s_1s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} &\sim s_1s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & 1 & \\ & 1 & 1 & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ &\sim s_1s_2 \begin{bmatrix} 1 & & y+1 & \\ & 1 & x+1 & y+1 \\ & & 1 & \\ & & & 1 \end{bmatrix}. \end{aligned}$$

Let x_1, x_2, x_3 and p be pairwise co-prime. Also assume y_1, y_2, y_3 and p to be pairwise co-prime. Let $x = {}^{x_3 p^{r_1}}/_{x_2}$ with $r_1 > 0$. Then the above calculation shows that we can change x to $x + 1 = {}^{(x_3 p^{r_1} + x_2)}/_{x_2}$. So we can always assume x to be of the form ${}^{x_1}/_{x_2 p^r}$ for some $r \geq 0$. Similarly, we can also assume y to be of the form ${}^{y_1}/_{y_2 p^s}$ with $s \geq 0$. Next, suppose $\tau = \gcd(x_1, y_1) > 1$. Then replacing $x = {}^{x_1}/_{x_2 p^r}$ by $x + \tau_1$, with τ_1 being the largest factor of y_1 that is co-prime to τ, we can also assume that $\gcd(x_1, y_1) = 1$.

Now we consider all the different possibilities that may arise. First of all, it is clear that, if both x and y are in $\mathbb{Z}$, i.e., $x_2 = y_2 = 1$, $r = 0$, $s = 0$ then $Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1 s_2 \Gamma_0(p^n)$. Next, if $x \in \mathbb{Z}$ but $y \notin \mathbb{Z}$, then

$$
\begin{aligned}
Q(\mathbb{Q})g\Gamma_0(p^n) &= Q(\mathbb{Q})s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} s_1 s_2 \Gamma_0(p^n) \\
&= Q(\mathbb{Q})s_1 \begin{bmatrix} 1 & y^{-1} & & \\ & 1 & & \\ & & 1 & -y^{-1} \\ & & & 1 \end{bmatrix} s_1 s_2 \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ y^{-1} & & 1 & \\ & y^{-1} & & 1 \end{bmatrix} \Gamma_0(p^n) \\
&= Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \frac{y_2 p^s}{y_1} & & 1 & \\ & \frac{y_2 p^s}{y_1} & & 1 \end{bmatrix} \Gamma_0(p^n).
\end{aligned}
$$

We note that the third equality follows from the following matrix identity.

$$
\begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} = \begin{bmatrix} -y^{-1} & 1 & & \\ & y & & \\ & & y^{-1} & 1 \\ & & & -y \end{bmatrix} s_1 \begin{bmatrix} 1 & y^{-1} & & \\ & 1 & & \\ & & 1 & -y^{-1} \\ & & & 1 \end{bmatrix}.
$$

But, now from the Lemma 8.1.5 and 8.1.3 it follows that if $0 < s < n$ then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})\begin{bmatrix}1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1\end{bmatrix}\Gamma_0(p^n),$$

and if $s \geq n$ then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n).$$

Further, If $s = 0$ then from the Lemma 8.1.5 and 8.1.7 it follows that

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1s_2\Gamma_0(p^n),$$

which is one of the listed representative in the statement of the theorem. Therefore we are done in this case.

Now consider the case when $x \notin \mathbb{Z}$ and $y \in \mathbb{Z}$. Then we have,

$$\begin{aligned}
Q(\mathbb{Q})g\Gamma_0(p^n) &= Q(\mathbb{Q})s_1s_2\begin{bmatrix}1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1\end{bmatrix}\Gamma_0(p^n) \\
&= Q(\mathbb{Q})\begin{bmatrix}x & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & \frac{1}{x}\end{bmatrix}s_1s_2\begin{bmatrix}1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1\end{bmatrix}\Gamma_0(p^n) \\
&= Q(\mathbb{Q})\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ x^{-1} & & & 1\end{bmatrix}\begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}\Gamma_0(p^n) \\
&= Q(\mathbb{Q})\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 1 & \\ x^{-1} & & & 1\end{bmatrix}\Gamma_0(p^n).
\end{aligned}$$

Now it follows from Lemma 8.1.6 and 8.1.7 that if $r = 0$ then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1s_2\Gamma_0(p^n),$$

and if $r \geq n$ then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n).$$

Further, if $0 < r < n$ then Lemma 8.1.6 yields that

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ c_1 p^r & & & 1 \end{bmatrix} \Gamma_0(p^n),$$

for some integer c_1 such that $\gcd(c_1, p) = 1$. Then it follows from Lemma 8.1.8 that g lies in the same double coset as one of the elements listed in the statement of the theorem.

Next, suppose $x \notin \mathbb{Z}$ and $y \notin \mathbb{Z}$. If $s = r = 0$ then from Lemma 8.1.1 and Lemma 8.1.7 it follows that

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})s_1 s_2\Gamma_0(p^n).$$

Further it follows from Lemma 8.1.1 that if $s \leq r$ and $r \geq n$ then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n);$$

otherwise, if $s \leq r < n$ then

$$Q(\mathbb{Q})g\Gamma_0(p^n) = Q(\mathbb{Q})g_1(x_3, p, r)\Gamma_0(p^n)$$

for some non-zero integer x_3 co-prime to p. But then these cases have already been considered . Hence, we are left with the case when $s > r$ and then if $s \geq n$ from Lemma 8.1.1 we get

$$Q(\mathbb{Q})\, g\, \Gamma_0(p^n) = Q(\mathbb{Q})1\Gamma_0(p^n),$$

and we are done. Otherwise still assuming $s > r$ but $s < n$, we get

$$Q(\mathbb{Q})\, g\, \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ \eta_1 p^s & & 1 & \\ \eta_2 p^{-r+2\,s} & \eta_1 p^s & & 1 \end{bmatrix} \Gamma_0(p^n),$$

where $\eta_1, \eta_2 \in \mathbb{Z}$ and $\gcd(\eta_i, p) = 1$ for $i = 1, 2$. In view of Lemma 8.1.4 it further reduces to

$$Q(\mathbb{Q})\, g\, \Gamma_0(p^n) = Q(\mathbb{Q}) \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ \eta_2 p^{-r+2\,s} & p^s & & 1 \end{bmatrix} \Gamma_0(p^n).$$

Now, the result follows from Lemma 8.1.9 and we are done in this case as well.

Fourth Cell: Next we consider an element g from the fourth cell and let

$$g = s_1 s_2 s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

If $x \in \mathbb{Z}$ then

$$Q(\mathbb{Q}) g \Gamma_0(p^n) = Q(\mathbb{Q}) s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z + 2xy & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \Gamma_0(p^n),$$

and we are reduced to the case of the third cell. Therefore let us assume that $x \notin \mathbb{Z}$. If necessary on multiplication by a suitable matrix from right, we can assume that, $x = {}^{x_1}/_{p^r x_2}$, $y = {}^{y_1}/_{p^s y_2}$ and $z = {}^{z_1}/_{p^{r_1} z_2}$ where x_i, y_i, $z_i \in \mathbb{Z}$, for $i = 1, 2$; r, s, r_1 are non-negative integers, x_1, x_2, p are mutually co-prime integers; y_1, y_2, p are mutually co-prime integers and z_1, z_2, p are also mutually co-prime integers. We can further adjust x_1, y_1 and z_1 by multiplication by a proper matrix from the right such that any two non-zero elements selected from the set $\{x_1, y_1, z_1, x_2, y_2, z_2, p\}$ are mutually co-prime except, possibly, when both the chosen elements belong to $\{x_2,\ y_2,\ z_2\}$. Then by the virtue of Lemma 8.1.2 once again we are reduced to the case of the third cell. This proves that the representatives listed in the theorem constitute a complete set of double coset representatives.

Disjointness.

Now we prove that the listed representatives in the theorem are disjoint. Let w_1 and w_2 be any two of the listed representatives. It is clear that w_1 and w_2 represent the same double coset if and only if there exists an element

$$q = \begin{bmatrix} t & & & \\ & a & b & \\ & c & d & \\ & & & (ad-bc)t^{-1} \end{bmatrix} \begin{bmatrix} 1 & l & \mu & k \\ & 1 & & \mu \\ & & 1 & -l \\ & & & 1 \end{bmatrix} \in Q(\mathbb{Q})$$

such that $w_2^{-1} q w_1 \in \Gamma_0(p^n)$. On comparing the multiplier on both the sides we conclude that $ad - bc = 1$. On explicitly writing $w_2^{-1} q w_1$ for each possible choice of w_1 and w_2 it becomes clear that both t and t^{-1} must be integers for the condition $w_2^{-1} q w_1 \in \Gamma_0(p^n)$ to hold. Therefore $t = \pm 1$. We can assume that $t = 1$. Also clearly q must be a matrix with integral entries. Now we consider different pairs of the listed representatives for checking disjointness.

$\underline{w_1 = g_3(p, \alpha, r, s),\ w_2 = g_3(p, \beta, v, w)}$. Let

$$w_1 = g_3(p, \alpha, r, s) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ \alpha p^r & p^s & & 1 \end{bmatrix} \text{ and } w_2 = g_3(p, \beta, v, w) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^w & & 1 & \\ \beta p^v & p^w & & 1 \end{bmatrix},$$

with α, β integers co-prime to p and $r, s, v, w \in \mathbb{Z}$ such that $0 < s < r < 2s, 0 < v < w < 2v, 0 < s, r < n, 0 < w, v < n$. We see that,

$$w_2^{-1} q w_1 = \begin{bmatrix} & & & * \\ & & & * \\ & & & -(dl - c\mu + kp^w)\alpha p^r - (\mu p^w - d)p^s - p^w \\ & & & -(\beta k p^v - blp^w + a\mu p^w - 1)\alpha p^r - (\beta \mu p^v + bp^w)p^s - \beta p^v \end{bmatrix}$$

$$\begin{matrix} & * & * & * \\ & * & * & * \\ -(dl-c\mu+kp^w)p^s-lp^w+c & * & * \\ -\beta lp^v-(\beta kp^v-blp^w+a\mu p^w-1)p^s-ap^w & * & * \end{matrix}\Bigg]$$

Suppose $s > w$. If $w_2^{-1}qw_1 \in \Gamma_0(p^n)$ then looking at the bottom two entries of the second column we conclude that p must divide both a and c. But, then it contradicts that $ad - bc = 1$. Similarly, if $s < w$ by looking at first two entries of the third row we get that $p|d$ and $p|c$ contradicting $ad - bc = 1$. Therefore, we assume $s = w$. Now looking at the bottom most entry of the first column we conclude that if $r \neq v$ then the valuation of this element can not be n. Therefore, if $r \neq v$ or $s \neq w$ then $g_3(p, \alpha, r, s)$ and $g_3(p, \beta, v, w)$ lie in different double cosets. If $r = v$ and $s = w$ then Lemma 8.1.9 describes the condition for $g_3(p, \alpha, r, s)$ and $g_3(p, \beta, v, w)$ to lie in the same double coset. We conclude that such representatives listed in the theorem represent disjoint double cosets.

$\underline{w_1 = g_3(p, \alpha, r, s),\ w_2 = g_2(p, w)}$. Let $w_1 = g_3(p, \alpha, r, s)$ and $w_2 = g_2(p, w)$. Assume $w_2^{-1}qw_1 \in \Gamma_0(p^n)$. Then we see that $w_2^{-1}qw_1 =$

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -(dl-c\mu+kp^w)\alpha p^r-(\mu p^w-d)p^s-p^w & -(dl-c\mu+kp^w)p^s-lp^w+c & * & * \\ (blp^w-a\mu p^w+1)\alpha p^r-bp^sp^w & (blp^w-a\mu p^w+1)p^s-ap^w & * & * \end{bmatrix}$$

and it is clear that $p|c$ and if $s > w$ then and $p|a$ or else if $s < w$ then $p|d$. In any case $p|ad - bc = 1$ which is a contradiction. Hence, we further assume $s = w$. Now, as $s < r < 2s$, looking at the last entry of the first column we see that the valuation of the element $(blp^s - a\mu p^s + 1)\alpha p^r - b(p^s)^2$ is r. Since $r < n$, we conclude that $g_3(p, \alpha, r, s)$ and $g_2(p, w)$ lie in different double cosets.

$\underline{w_1 = g_3(p, \alpha, r, s),\ w_2 = g_1(p, \beta, v)}$. Let $w_1 = g_3(p, \alpha, r, s)$ and $w_2 = g_1(p, \beta, v)$. Assume $w_2^{-1}qw_1 \in \Gamma_0(p^n)$. Then we see that,

$$w_2^{-1}qw_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ -(dl-c\mu)\alpha p^r + dp^s & -(dl-c\mu)p^s + c & * & * \\ -\beta\mu p^s p^v - (\beta k p^v - 1)\alpha p^r - \beta p^v & -\beta l p^v - (\beta k p^v - 1)p^s & * & * \end{bmatrix}.$$

Clearly, $p|c$. Since, $r > s$, p also divides d and it contradicts the condition $ad - bc = 1$. Therefore $g_3(p, \alpha, r, s)$ and $w_2 = g_1(p, \beta, v)$ lie in different double cosets.

$w_1 = g_2(p, s)$, $w_2 = g_1(p, \beta, v)$. Let $w_1 = g_2(p, s)$ and $w_2 = g_1(p, \beta, v)$. Assume that $w_2^{-1}qw_1 \in \Gamma_0(p^n)$. Then we see that,

$$w_2^{-1}qw_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ dp^s & -(dl-c\mu)p^s + c & * & * \\ -\beta\mu p^s p^v - \beta p^v & -\beta l p^v - (\beta k p^v - 1)p^s & * & * \end{bmatrix}.$$

Once again we see that p divides both c and d which is a contradiction to the condition $ad - bc = 1$. Therefore $g_2(p, s)$ and $w_2 = g_1(p, \beta, v)$ lie in different double cosets.

$w_1 = g_2(p, s)$, $w_2 = g_2(p, w)$. Let $w_1 = g_2(p, s)$ and $w_2 = g_1(p, w)$. Let us assume that $w_2^{-1}qw_1 \in \Gamma_0(p^n)$. Then we see that,

$$w_2^{-1}qw_1 = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ dp^s & -(dl-c\mu)p^s + c & * & * \\ -\beta\mu p^s p^w - \beta p^w & -\beta l p^w - (\beta k p^w - 1)p^s & * & * \end{bmatrix}.$$

Once again we see that $p|c$ and if $s > w$ then $p|a$ or else if $s < w$ then $p|d$. In any case $p|ad - bc = 1$, which is a contradiction. Therefore $g_2(p, s)$ and $g_2(p, w)$ lie in different double cosets.

$w_1 = g_1(p, \alpha, r)$, $w_2 = g_1(p, \beta, v)$. Let $w_1 = g_1(p, \alpha, r)$ and $w_2 = g_1(p, \beta, v)$. Let us assume that $w_2^{-1} q w_1 \in \Gamma_0(p^n)$. Then we see that,

$$w_2^{-1} q w_1 = \begin{bmatrix} & * & * & * & * \\ & * & * & * & * \\ & -(dl - c\mu)\alpha p^r & c & * & * \\ -(\beta k p^v - 1)\alpha p^r - \beta p^v & -\beta l p^v & * & * \end{bmatrix}.$$

Since $r, v < n$, we see that if $r \neq v$ then valuation of $-(\beta k p^v - 1)\alpha p^r - \beta p^v$ is less than n. Therefore $g_1(p, \alpha, r)$ and $g_1(p, \beta, v)$ lie in different double cosets. This completes the proof of disjointness.

The number of representatives.

Finally, we calculate the total number of inequivalent representatives. First let n be even, say $n = 2m$ for some positive integer m. Then

$$\begin{aligned}
&\#(Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(p^{2m})) \\
&= 2 + 2m - 1 + \sum_{r=1}^{2m-1} \phi(p^{\min(r, 2m-r)}) + \sum_{s=1}^{2m-1} \sum_{r=s+1}^{\min(2s-1, 2m-1)} \phi(p^{\min(2s-r, 2m-r)}) \\
&= \frac{p^{m+1} + p^m - 2}{p - 1}.
\end{aligned}$$

Similarly, if n is odd, say $n = 2m + 1$ then,

$$\begin{aligned}
&\#(Q(\mathbb{Q})\backslash \mathrm{GSp}(4, \mathbb{Q})/\Gamma_0(p^{2m+1})) \\
&= 2 + 2m + \sum_{r=1}^{2m} \phi(p^{\min(r, 2m+1-r)}) + \sum_{s=1}^{2m} \sum_{r=s+1}^{\min(2s-1, 2m)} \phi(p^{\min(2s-r, 2m+1-r)}) \\
&= 2 + 2m + \sum_{r=1}^{m} \phi(p^r) + \sum_{r=m+1}^{2m} \phi(p^{2m+1-r}) + \sum_{s=1}^{m} \sum_{r=s+1}^{2s-1} \phi(p^{2s-r}) \\
&+ \sum_{s=m+1}^{2m} \sum_{r=s+1}^{2m} \phi(p^{2m+1-r}) \\
&= 2 + 2m + p^m - 1 + p^m - 1 + \frac{p^m - mp + m - 1}{p - 1} + \frac{p^m - mp + m - 1}{p - 1}
\end{aligned}$$

$$= \frac{2(p^{m+1} - mp + m - 1)}{p-1} = \frac{2(p^{m+1} - 1)}{p-1}.$$

Thus on combining these we obtain the formula (8.13) giving the number of one-dimensional cusps. □

8.2 Double coset decomposition $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N)$

We begin by proving the following lemma.

Lemma 8.2.1. *Assume $N = \prod_{i=1}^{m} p_i^{n_i}$. Let $C_1(p_i^{n_i})$ denote the number of inequivalent representatives for the double cosets $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p_i^{n_i})$ (refer Theorem 8.1.10). Then, the number of inequivalent representatives for the double cosets $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(\mathrm{N})$ is given by $C_1(N) = \prod_{i=1}^{m} C_1(p_i^{n_i})$.*

Proof. Before proceeding with the proof we recall $\Gamma_\infty(\mathbb{Z}) := Q(\mathbb{Q}) \cap \mathrm{Sp}(4,\mathbb{Z})$, $\Delta(\mathbb{Z}/N\mathbb{Z}) := \{\left[\begin{smallmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{smallmatrix}\right] \in \mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})\}$ and $\Gamma_\infty(\mathbb{Z}/N\mathbb{Z}) := \{\gamma \mod N \mid \gamma \in \Gamma_\infty(\mathbb{Z})\}$.

Next we note that the representatives for $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N)$ may be obtained from the representatives of $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(p_i^{n_i})$ for $i = 1$ to m. This observation is essentially based on the following two well known facts.

1. The natural projection map from $\mathrm{Sp}(4,\mathbb{Z})$ to $\mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})$ is surjective.

2. $\mathrm{Sp}(4,\mathbb{Z}/\prod_p p^e\mathbb{Z}) \xrightarrow{\sim} \prod_p \mathrm{Sp}(4,\mathbb{Z}/p^e\mathbb{Z})$.

In fact, we have

$$\begin{aligned} \mathrm{Sp}(4,\mathbb{Z})/\Gamma_0(\mathrm{N}) &\xrightarrow{\sim} \left(\mathrm{Sp}(4,\mathbb{Z})/\Gamma(N)\right)/\left(\Gamma_0(\mathrm{N})/\Gamma(N)\right) \\ &\xrightarrow{\sim} \mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})/\Delta(\mathbb{Z}/N\mathbb{Z}). \end{aligned}$$

Clearly, $\Delta(\mathbb{Z}/N\mathbb{Z}) = \prod_{i=1}^m \Delta(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$ and the following diagram is commutative.

$$\begin{array}{ccc}
\mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z}) & \xrightarrow[\psi]{\sim} & \prod_{i=1}^m \mathrm{Sp}(4,\mathbb{Z}/p_i^{n_i}\mathbb{Z}) \\
\Cup & & \Cup \\
g & & (g_1,\cdots,g_m) \\
\downarrow & & \downarrow \\
g\Delta(\mathbb{Z}/N\mathbb{Z}) & & (g_1\Delta(\mathbb{Z}/p_1^{n_1}\mathbb{Z}),\cdots,g_m\Delta(\mathbb{Z}/p_m^{n_m}\mathbb{Z})) \\
\Cap & & \Cap \\
A = \mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})/\Delta(\mathbb{Z}/N\mathbb{Z}) & \xrightarrow[\phi]{\sim} & \prod_{i=1}^m \mathrm{Sp}(4,\mathbb{Z}/p_i^{n_i}\mathbb{Z})/\Delta(\mathbb{Z}/p_i^{n_i}\mathbb{Z}) = B
\end{array}$$

Next we show that the left action by $\Gamma_\infty(\mathbb{Z})$ is compatible with the isomorphisms described in the commutative diagram above. In fact, $\Gamma_\infty(\mathbb{Z})$ acts on both sides as follows:

- on A: via

$$\Gamma_\infty(\mathbb{Z}) \to \Gamma_\infty(\mathbb{Z}/N\mathbb{Z})$$
$$\gamma \to \bar{\gamma}$$

- on B: via

$$\Gamma_\infty(\mathbb{Z}) \to \Gamma_\infty(\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\sim} \prod_{i=1}^m \Gamma_\infty(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$$
$$\gamma \to \bar{\gamma} \qquad \xrightarrow{\sim} (\gamma_1,\gamma_2,\cdots\cdots\gamma_{m-1},\gamma_m)\,.$$

Let $g \in \mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})$, $a = g\Delta(\mathbb{Z}/N\mathbb{Z}) \in A$ and $\gamma \in \Gamma_\infty(\mathbb{Z})$ then it is easy to check that $\phi(\gamma a) = \gamma(\phi(a))$. Therefore we obtain,

$$\begin{aligned} Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N) &\tilde{\to} (Q(\mathbb{Q}) \cap \mathrm{Sp}(4,\mathbb{Z}))\backslash\mathrm{Sp}(4,\mathbb{Z})/\Gamma_0(\mathrm{N}) \\ &\tilde{\to} \Gamma_\infty(\mathbb{Z}/N\mathbb{Z})\backslash\mathrm{Sp}(4,\mathbb{Z}/N\mathbb{Z})/\Delta(\mathbb{Z}/N\mathbb{Z}) \\ &\tilde{\to} \prod_{i=1}^{m} \Gamma_\infty(\mathbb{Z}/p_i^{n_i}\mathbb{Z})\backslash\mathrm{Sp}(4,\mathbb{Z}/p_i^{n_i}\mathbb{Z})/\Delta(\mathbb{Z}/p_i^{n_i}\mathbb{Z}). \end{aligned}$$

Now, the result follows from Theorem 8.1.10. □

Now we describe a minimal and complete set of representatives for the double cosets $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(\mathrm{N})$ in the following theorem.

Theorem 8.2.2. *Assume* $N = \prod_{i=1}^{m} p_i^{n_i}$. *A complete and minimal system of representatives for the double cosets* $Q(\mathbb{Q})\backslash\mathrm{GSp}(4,\mathbb{Q})/\Gamma_0(N)$ *is given by*

$$\begin{aligned} g_1(\gamma, x) &= \begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ x\gamma &&& 1 \end{bmatrix}, \ 1 \le \gamma \le N,\ \gamma|N, \\ g_3(\gamma,\delta,y) &= \begin{bmatrix} 1 &&& \\ & 1 && \\ \delta && 1 & \\ y\gamma & \delta && 1 \end{bmatrix}, \ 1 < \delta < \gamma \le N,\ \gamma|N,\ \delta|N,\ \delta|\gamma,\ \gamma|\delta^2; \end{aligned}$$

where for fixed γ *and* δ *we have*

$$x = M + \zeta \prod_{p_i \nmid M,\, p_i|N} p_i^{n_i},\ y = L + \theta \prod_{p_i \nmid L,\, p_i|N} p_i^{n_i}$$

with $M = \mathrm{gcd}(\gamma, \frac{N}{\gamma})$, $L = \mathrm{gcd}(\frac{\delta^2}{\gamma}, \frac{N}{\gamma})$, ζ *and* θ *varies through all the elements of* $(\mathbb{Z}/M\mathbb{Z})^\times$ *and* $(\mathbb{Z}/L\mathbb{Z})^\times$ *respectively. Here we interpret* $(\mathbb{Z}/\mathbb{Z})^\times$ *as an empty set.*

Proof. It is easy to check that the listed representatives are disjoint. Let $\alpha(N)$ denote the total number of representatives listed in the statement of the theorem. We note that for $N = p^n$, with p a prime and $n \ge 1$, the number of listed representatives are the same as given by Lemma 8.1.10 (moreover, the set of representatives in this case will be seen to be equivalent to the set of representatives given by Theorem 8.1.10 if one applies the second part of Lemma 8.1.9 and works out the details). We will show that for any pair of co-prime positive integers R and S we have $\alpha(RS) = \alpha(R)\alpha(S)$. Then it will follow that the listed representatives form a complete set because for any N their number will agree with the number given in Lemma 8.2.1. We have,

$$\begin{aligned}
\alpha(RS) &= 1 + \sum_{\substack{\gamma|RS \\ 1<\gamma\leq RS}} \phi(\gcd(\gamma, \frac{RS}{\gamma})) + \sum_{\substack{\gamma|RS \\ 1<\gamma\leq RS}} \sum_{\substack{\delta|\gamma,\, \gamma|\delta^2 \\ \gamma>\delta}} \phi(\gcd(\frac{\delta^2}{\gamma}, \frac{RS}{\gamma})) \\
&= 1 + \sum_{\substack{\gamma|RS \\ 1<\gamma\leq RS}} \sum_{\substack{\delta|\gamma,\, \gamma|\delta^2 \\ \gamma\geq\delta}} \phi(\gcd(\frac{\delta^2}{\gamma}, \frac{RS}{\gamma})) \\
&= 1 + \sum_{\substack{\gamma_1|R,\, \gamma_2|S \\ 1<\gamma_1\gamma_2\leq RS}} \sum_{\substack{\delta|\gamma_1\gamma_2,\, \gamma_1\gamma_2|\delta^2 \\ \gamma_1\gamma_2\geq\delta}} \phi(\gcd(\frac{\delta^2}{\gamma_1\gamma_2}, \frac{RS}{\gamma_1\gamma_2})) \\
&\quad + \sum_{\substack{\gamma_1|R \\ 1<\gamma_1\leq R}} \sum_{\substack{\delta|\gamma_1,\, \gamma_1|\delta^2 \\ \gamma_1\geq\delta}} \phi(\gcd(\frac{\delta^2}{\gamma_1}, \frac{RS}{\gamma_1})) \sum_{\substack{\gamma_2|S \\ 1<\gamma_1\leq S}} \sum_{\substack{\delta|\gamma_2,\, \gamma_2|\delta^2 \\ \gamma_2\geq\delta}} \phi(\gcd(\frac{\delta^2}{\gamma_2}, \frac{RS}{\gamma_2})) \\
&= \left(1 + \sum_{\substack{\gamma_1|R \\ 1<\gamma_1\leq R}} \sum_{\substack{\delta_1|\gamma_1,\, \gamma_1|\delta_1^2 \\ \gamma_1\geq\delta_1}} \phi(\gcd(\frac{\delta_1^2}{\gamma_1}, \frac{R}{\gamma_1}))\right) \\
&\qquad\qquad \left(1 + \sum_{\substack{\gamma_2|S \\ 1<\gamma_2\leq S}} \sum_{\substack{\delta_2|\gamma_2,\, \gamma_2|\delta_2^2 \\ \gamma_2\geq\delta_2}} \phi(\gcd(\frac{\delta_2^2}{\gamma_2}, \frac{S}{\gamma_2}))\right) \\
&= \alpha(R)\alpha(S).
\end{aligned}$$

This completes the proof. $\square$

8.3 Double coset decomposition $Q(\mathbb{Q}_p)\backslash\mathrm{GSp}(4,\mathbb{Q})/\mathrm{Si}(p^n)$

We thank Prof. Ralf Schmidt and Prof. Brooks Roberts for sharing the following result from one of their forthcoming manuscripts.

Proposition 8.3.1. *Assume $n \geq 1$. A complete and minimal system of representatives for the double cosets $Q(\mathbb{Q}_p)\backslash\mathrm{GSp}(4,\mathbb{Q}_p)/\mathrm{Si}(p^n)$ is given by*

$$1, \qquad s_1s_2, \qquad \begin{bmatrix}1&&&\\&1&&\\&&1&\\p^r&&&1\end{bmatrix},\ 1\leq r\leq n-1, \qquad \begin{bmatrix}1&&&\\&1&&\\p^s&&1&\\&p^s&&1\end{bmatrix},\ 1\leq s\leq n-1,$$

$$\begin{bmatrix}1&&&\\&1&&\\p^s&&1&\\p^r&p^s&&1\end{bmatrix},\ 1\leq s,r\leq n-1,\ s<r<2s.$$

In particular, $\#(Q(\mathbb{Q}_p)\backslash\mathrm{GSp}(4,\mathbb{Q}_p)/\mathrm{Si}(p^n)) = \left[\dfrac{(n+2)^2}{4}\right]$ *for all $n\geq 0$.*

Proof. We first prove disjointness. Abbreviate

$$g_r = \begin{bmatrix}1&&&\\&1&&\\&&1&\\p^r&&&1\end{bmatrix}, \qquad h_s = \begin{bmatrix}1&&&\\&1&&\\p^s&&1&\\&p^s&&1\end{bmatrix}.$$

Two elements A and B represent the same double coset if and only if there exists an element

$$q = \begin{bmatrix}e&&&\\&a&b&\\&c&d&\\&&&(ad-bc)e^{-1}\end{bmatrix}\begin{bmatrix}1&w&&\\&1&&\\&&1&-w\\&&&1\end{bmatrix}\begin{bmatrix}1&&y&z\\&1&&y\\&&1&\\&&&1\end{bmatrix} \in Q(\mathbb{Q}_p)$$

such that $A^{-1}qB \in \mathrm{Si}(p^n)$. If we have this relation where A and B are two of the listed elements, then necessarily $q \in Q(\mathbb{Z}_p)$. Since the multiplier on both sides must be a unit, we conclude that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z}_p)$. Then also $e \in \mathbb{Z}_p^\times$. Multiplying the relation $A^{-1}qB \in \mathrm{Si}(p^n)$ from the left with $e^{-1}1$, we may assume that $e = 1$. We go through the cases:

<u>g_r and g_t</u>. We have

$$g_t^{-1} q g_r = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ p^t + p^r(ad - bc - p^t(wy + z)) & * & * & * \end{bmatrix}.$$

Since $ad - bc$ is a unit, the valuation of $p^r(ad - bc - p^t(wy + z))$ is r. Hence this matrix can only then be in $\mathrm{Si}(p^n)$ if $r = t$.

<u>g_r and h_s</u>. We have

$$h_s^{-1} q g_r = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & c - wp^s & * & * \\ * & -ap^s & * & * \end{bmatrix}.$$

From $c - wp^s \in p^n\mathbb{Z}_p$ it follows that c is not a unit. It then follows from $ad - bc \in \mathbb{Z}_p^\times$ that a is a unit. Hence $-ap^s \notin p^n\mathbb{Z}_p$.

<u>g_r and $g_t h_s$</u>. We have

$$(g_t h_s)^{-1} q g_r = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & c - wp^s & * & * \\ * & -ap^s - wp^t & * & * \end{bmatrix}.$$

From $c - wp^s \in p^n\mathbb{Z}_p$ it follows that c is not a unit, and then $a \in \mathbb{Z}_p^\times$. Since $s < t$, we get $v(-ap^s - wp^t) = s$. Hence the matrix cannot be in $\mathrm{Si}(p^n)$.

<u>h_s and h_u</u>. We have

$$h_u^{-1} q h_s = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ dp^s - p^u(1 + p^s y) & c + p(\ldots) & * & * \\ * & * & * & * \end{bmatrix}.$$

If this matrix is in $\mathrm{Si}(p^n)$, then c cannot be a unit. But $ad - bc \in \mathbb{Z}_p^\times$, hence

a and d are units. Therefore $v(dp^s) = s$ and $v(p^u(1+p^s y)) = u$. Hence from $dp^s - p^u(1+p^s y) \in p^n\mathbb{Z}_p$ it follows that $u = s$.

$\underline{h_s \text{ and } g_r h_u}$. We have

$$(g_r h_u)^{-1} q h_s = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ dp^s - p^u(1+p^s y) & c + p(\ldots) & * & * \\ * & * & * & * \end{bmatrix}.$$

As in the previous case we conclude $u = s$. But

$$(g_r h_s)^{-1} q h_s = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ -p^r - bp^{2s} - yp^{r+s} & * & * & * \end{bmatrix}.$$

Since $r < 2s$ we get $v(-p^r - bp^{2s} - yp^{r+s}) = r$. Hence the matrix cannot be in $\mathrm{Si}(p^n)$.

$\underline{g_r h_s \text{ and } g_t h_u}$. We have

$$(g_t h_u)^{-1} q g_r h_s = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & c + p(\ldots) & * & * \\ * & -ap^u - wp^t + p^s(ad - bc + p(\ldots)) & * & * \end{bmatrix}.$$

Assume this matrix is in $\mathrm{Si}(p^n)$. It follows from $c + p(\ldots) \in p^n\mathbb{Z}_p$ that c cannot be a unit. Since $ad - bc \in \mathbb{Z}_p^\times$, we get that a is a unit (also d). Since $u < t$ we obtain $v(-ap^u - wp^t) = u$. But $v(p^s(ad - bc + p(\ldots))) = s$. Hence the $(2,4)$–coefficient can only then be in $p^n\mathbb{Z}_p$ if $u = s$. Now

$$(g_t h_s)^{-1} q g_r h_s = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ p^r(ad - bc) - bp^{2s} + (bw - ay)p^{r+s} - p^t(1 + p(\ldots)) & * & * & * \end{bmatrix}.$$

Since $r < 2s$, we have $v(p^r(ad-bc) - bp^{2s} + (bw-ay)p^{r+s}) = r$. But $v(p^t(1+p(\ldots))) = t$. It follows that the $(4,1)$–coefficient can only then be in $\mathrm{Si}(p^n)$ if $t = r$. We proved that $u = s$ and $t = r$, so the two representatives are identical.

In a similar manner one can check all matrices against the representatives 1 and $s_1 s_2$. The result is that all the double cosets represented by the elements listed in the proposition are pairwise disjoint.

Next we prove that the double cosets represented by the listed elements exhaust all of $\mathrm{GSp}(4, \mathbb{Q}_p)$. By the Bruhat decomposition,

$$\begin{aligned}\mathrm{GSp}(4, \mathbb{Q}_p) =& Q(\mathbb{Q}_p) \sqcup Q(\mathbb{Q}_p) s_1 \begin{bmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{bmatrix} \sqcup Q(\mathbb{Q}_p) s_1 s_2 \begin{bmatrix} 1 & & * & \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} \\ & \sqcup Q(\mathbb{Q}_p) s_1 s_2 s_1 \begin{bmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}. \end{aligned} \tag{8.15}$$

If $g \in Q(\mathbb{Q}_p)$, then, of course, $Q(\mathbb{Q}_p) g \mathrm{Si}(p^n)$ is represented by 1. Assume that g is in the second cell. We may assume that

$$g = s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix}, \qquad x \in \mathbb{Q}_p.$$

If $x \in \mathbb{Z}_p$, then $Q(\mathbb{Q}_p) g \mathrm{Si}(p^n) = Q(\mathbb{Q}_p) s_1 \mathrm{Si}(p^n) = Q(\mathbb{Q}_p) 1 \mathrm{Si}(p^n)$. But if $x \notin \mathbb{Z}_p$, then, using the usual matrix identity,

$$\begin{aligned} Q(\mathbb{Q}_p) g \mathrm{Si}(p^n) &= Q(\mathbb{Q}_p) \begin{bmatrix} 1 & & & \\ x & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} s_1 \mathrm{Si}(p^n) \\ &= Q(\mathbb{Q}_p) s_1 \begin{bmatrix} 1 & x^{-1} & & \\ & 1 & & \\ & & 1 & -x^{-1} \\ & & & 1 \end{bmatrix} s_1 \mathrm{Si}(p^n) = Q(\mathbb{Q}_p) 1 \mathrm{Si}(p^n). \end{aligned}$$

Now let g be an element in the third cell. We may assume that

$$g = s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}, \qquad x, y \in \mathbb{Q}_p.$$

If both x and y are in $\mathbb{Z}_p$, then $Q(\mathbb{Q}_p)g\mathrm{Si}(p^n) = Q(\mathbb{Q}_p)s_1s_2\mathrm{Si}(p^n)$. If $x \in \mathbb{Z}_p$ but $y \notin \mathbb{Z}_p$, then

$$\begin{aligned} Q(\mathbb{Q}_p)g\mathrm{Si}(p^n) &= Q(\mathbb{Q}_p)s_1s_2 \begin{bmatrix} 1 & & y & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) = Q(\mathbb{Q}_p) \begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & \\ & & -y & 1 \end{bmatrix} s_1s_2\mathrm{Si}(p^n) \\ &= Q(\mathbb{Q}_p)s_1 \begin{bmatrix} 1 & y^{-1} & & \\ & 1 & & \\ & & 1 & -y^{-1} \\ & & & 1 \end{bmatrix} s_1s_2\mathrm{Si}(p^n) \\ &= Q(\mathbb{Q}_p) \begin{bmatrix} 1 & & & \\ & 1 & & \\ y^{-1} & & 1 & \\ & y^{-1} & & 1 \end{bmatrix} \mathrm{Si}(p^n). \end{aligned}$$

Since $y^{-1} \in p\mathbb{Z}_p$, the last matrix is equivalent to one of our representatives. Now assume $x \notin \mathbb{Z}_p$ but $y \in \mathbb{Z}_p$. Then

$$\begin{aligned} Q(\mathbb{Q}_p)g\mathrm{Si}(p^n) &= Q(\mathbb{Q}_p)s_1s_2 \begin{bmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) = Q(\mathbb{Q}_p) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x & & & 1 \end{bmatrix} s_1s_2\mathrm{Si}(p^n) \\ &= Q(\mathbb{Q}_p)s_1s_2s_1 \begin{bmatrix} 1 & & & x^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1s_2\mathrm{Si}(p^n) \\ &= Q(\mathbb{Q}_p) \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x^{-1} & & & 1 \end{bmatrix} \mathrm{Si}(p^n). \end{aligned}$$

Since $x^{-1} \in p\mathbb{Z}_p$, the last matrix is equivalent to one of our representatives. Now assume that $x \notin \mathbb{Z}_p$ and $y \notin \mathbb{Z}_p$. Then

$$\begin{aligned}
Q(\mathbb{Q}_p)g\mathrm{Si}(p^n) &= Q(\mathbb{Q}_p)\begin{bmatrix} 1 & & & \\ y & 1 & & \\ & & 1 & \\ x & & -y & 1 \end{bmatrix} s_1 s_2 \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)s_1 \begin{bmatrix} 1 & y^{-1} & & \\ & 1 & & \\ & & 1 & -y^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} -x^{-1} & & & -1 \\ & 1 & & \\ & & 1 & \\ & & & -x \end{bmatrix} s_1 s_2 s_1 \begin{bmatrix} 1 & & & x^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)\begin{bmatrix} 1 & & & \\ y^{-1} & 1 & & \\ & & 1 & \\ & & -y^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -x^{-1} & & \\ & & -x & \\ & & & 1 \end{bmatrix} s_2 s_1 \begin{bmatrix} 1 & & & x^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1 s_2 \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)\begin{bmatrix} 1 & & & \\ -xy^{-1} & 1 & & \\ & & 1 & \\ & & xy^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & x^{-1} & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n). \qquad (8.16)
\end{aligned}$$

Assume that $xy^{-1} \notin \mathbb{Z}_p$. Then

$$\begin{aligned}
Q(\mathbb{Q}_p)g\mathrm{Si}(p^n) &= Q(\mathbb{Q}_p)s_1 \begin{bmatrix} 1 & -x^{-1}y & & \\ & 1 & & \\ & & 1 & x^{-1}y \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & x^{-1} & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)s_1 \begin{bmatrix} 1 & & & \\ & 1 & & \\ & x^{-1} & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ x^{-1} & & & 1 \end{bmatrix} \mathrm{Si}(p^n).
\end{aligned}$$

Since $x^{-1} \in p\mathbb{Z}_p$, this is one of our cosets. But if $xy^{-1} \in \mathbb{Z}_p$, then

$$\begin{aligned}
Q(\mathbb{Q}_p)g\mathrm{Si}(p^n) &= Q(\mathbb{Q}_p)\begin{bmatrix} 1 & & & \\ -xy^{-1} & 1 & & \\ & & 1 & \\ & & xy^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & x^{-1} & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)\begin{bmatrix} 1 & & & \\ & 1 & & \\ y^{-1} & x^{-1} & 1 & \\ xy^{-2} & y^{-1} & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -xy^{-1} & 1 & & \\ & & 1 & \\ & & xy^{-1} & 1 \end{bmatrix} \mathrm{Si}(p^n)
\end{aligned}$$

$$= Q(\mathbb{Q}_p) \begin{bmatrix} 1 & & & \\ & 1 & & \\ y^{-1} & & 1 & \\ xy^{-2} & y^{-1} & & 1 \end{bmatrix} \mathrm{Si}(p^n).$$

Since $xy^{-1} \in \mathbb{Z}_p$, we are in a situation

$$Q(\mathbb{Q}_p) g \mathrm{Si}(p^n) = Q(\mathbb{Q}_p) \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ p^r & p^s & & 1 \end{bmatrix} \mathrm{Si}(p^n) \qquad \text{with } 0 < s \leq r.$$

We are done if $r \geq n$. Assume that $0 < s \leq r < n$. If $s = r$, then we are also done because of

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ p^r & & 1 & \\ p^r & p^r & & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & & \\ & 1 & & \\ & -p^r & 1 & 1 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ p^r & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}. \tag{8.17}$$

Assume therefore that $0 < s < r < n$. If $r \geq 2s$, then we are done again because of

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ p^r & p^s & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & -p^{r-2s} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ p^s & & 1 & \\ & p^s & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ p^{r-s} & 1 & p^{r-2s} & \\ & & 1 & \\ & & -p^{r-s} & 1 \end{bmatrix}. \tag{8.18}$$

We can therefore assume that $0 < s, r < n$ and $s < r < 2s$. But then we have one of the representatives listed in the proposition. Finally, assume that an element

$$g = s_1 s_2 s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

in the last cell in the decomposition (8.15) is given. If $x \in \mathbb{Z}_p$, then

$$Q(\mathbb{Q}_p) g \mathrm{Si}(p^n) = Q(\mathbb{Q}_p) s_1 s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z + 2xy & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n),$$

and we are reduced to the case of the third cell. If $x \notin \mathbb{Z}_p$, then

$$
\begin{aligned}
Q(\mathbb{Q}_p)g\mathrm{Si}(p^n) &= Q(\mathbb{Q}_p)s_1s_2s_1 \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p) \begin{bmatrix} 1 & & & \\ & 1 & & \\ x & & 1 & \\ & x & & 1 \end{bmatrix} s_1s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)s_2s_1s_2 \begin{bmatrix} 1 & & x^{-1} & \\ & 1 & & x^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} s_1s_2 \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)s_2s_1s_2s_1s_2 \begin{bmatrix} 1 & x^{-1} & & \\ & 1 & & \\ & & 1 & -x^{-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & \\ & 1 & z & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)s_1s_2s_1 \begin{bmatrix} 1 & & y' & z' \\ & 1 & x' & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n) \\
&= Q(\mathbb{Q}_p)s_1s_2 \begin{bmatrix} 1 & & y' & \\ & 1 & z' & y' \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathrm{Si}(p^n)
\end{aligned}
$$

with new elements x', y', z'. Hence we are again reduced to the case of the third cell. This completes the proof. $\square$

Chapter 9

Co-dimensions of the spaces of cusp forms

9.1 Introduction

In this chapter first we will briefly review Satake compactification and cusps and then prove the main result of this chapter, Theorem 9.4.1. We will also prove Theorem 9.3.1 giving a classical construction of a linearly independent set of Klingen Eisenstein series with respect to $\Gamma_0(N)$. We note that in this chapter we will use the classical version of GSp(4).

9.2 Cusps of $\Gamma_0(N)$

We recall a few basic facts related to the Satake compactification $S(\Gamma\backslash\mathbb{H}_2)$ of $\Gamma\backslash\mathbb{H}_2$ (cf.[33],[34],[3],[29]). Here Γ is a congruence subgroup of $\mathrm{Sp}(4,\mathbb{Z})$. We will be interested in $S(N) := S(\Gamma_0(N)\backslash\mathbb{H}_2)$. By $\mathrm{Bd}(N)$ we denote the boundary of $S(N)$. The one-dimensional components of $\mathrm{Bd}(N)$ are modular curves and are called the one-dimensional cusps. The one-dimensional cusps intersect on the zero-dimensional cusps. We define $M_k(\mathrm{Bd}(N))$ to be the space of modular forms on $\mathrm{Bd}(N)$ which consists of modular forms of weight k on the one-dimensional boundary components such that they are compatible on each intersection point. In the following we make the above description more explicit. Let $\mathrm{GSp}(4,\mathbb{Q}) = \sqcup_{i=1}^{l}\Gamma_0(\mathrm{N})g_iQ(\mathbb{Q})$. Let $\omega_1(q) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ for $q = \begin{bmatrix} * & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & * \end{bmatrix} \in Q(\mathbb{Q})$ and let $\imath_1$ be the embedding map $\imath_1(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]) = \begin{bmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{bmatrix}$ from $\mathrm{SL}(2,\mathbb{Q})$ to $Q(\mathbb{Q})$. Then the one-dimensional cusps bijectively correspond to $\{g_i\}$. Let $\Gamma_i = \omega_1(g_i^{-1}\Gamma_0(\mathrm{N})g_i \cap Q(\mathbb{Q}))$. In this situation the one-dimensional cusp g_i can be associated to the modular curve $\Gamma_i\backslash\mathbb{H}_1$. The zero-dimensional cusps of $\Gamma_i\backslash\mathbb{H}_1$ correspond to the representatives h_j of $\Gamma_i\backslash\mathrm{SL}(2,\mathbb{Z})/\Gamma_\infty^2(\mathbb{Z})$. In fact, h_j can be identified with the zero-dimensional cusp of $S(N)$ that corresponds to $\Gamma_0(N)g_i\imath_1(h_j)P(\mathbb{Q})$. If $\Gamma_0(N)g_i\imath_1(h_j)P(\mathbb{Q}) = \Gamma_0(N)g_r\imath_1(h_j)P(\mathbb{Q})$ for two inequivalent one-dimensional cusps g_i and g_r then it means that these two one-dimensional cusps intersect at a zero-dimensional cusp. Next for $F \in M_k(\Gamma_0(N))$, we define a function $\Phi(F)$ on $\mathbb{H}_1$ by $(\Phi(F))(z) = \lim_{\lambda\to\infty} F\left(\left[\begin{smallmatrix} z & \\ & i\lambda \end{smallmatrix}\right]\right)$ with $z \in \mathbb{H}_1$. It is clear that $\Phi(F|_kg_i)$ defines a map from $M_k(\Gamma_0(N))$ to $M_k(\Gamma_i)$, where $|_k$ denotes the usual slash operator defined as $F|_kg = \det(CZ+D)^{-k}F(g\langle Z\rangle)$ for $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and $g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$ with $g \in \mathrm{Sp}(4,\mathbb{R})$. Then we define $\tilde{\Phi} : M_k(\Gamma_0(N)) \to M_k(\mathrm{Bd}(N))$ by $F \to (\Phi(F|_kg_i))_{1\le i\le l}$. Any element $(f_i)_{1\le i\le l}$ in the image of $\tilde{\Phi}$ satisfies the condition that $f_i|_kh_1 = f_j|_kh_2$ whenever $\Gamma_0(N)g_i\imath_1(h_1)P(\mathbb{Q}) = \Gamma_0(N)g_j\imath_1(h_2)P(\mathbb{Q})$; where $h_1, h_2 \in \mathrm{SL}(2,\mathbb{Q})$ and $1 \le i, j \le l$. It essentially means that f_i and f_j, which are modular forms on the one-dimensional cusps g_i and g_j respectively, are compatible on the intersection points of these cusps.

9.3 Klingen Eisenstein Series with respect to $\Gamma_0(N)$

Now we describe an explicit construction of a set of linearly independent Klingen Eisenstein Series with respect to $\Gamma_0(N)$.

Theorem 9.3.1. *Assume $N \geq 1$. Let $g_1(\gamma, x)$ and $g_3(\gamma, \delta, y)$ be as in Theorem 8.2.2.*

1. *Let f_1 be an elliptic cusp form of even weight k with $k \geq 6$ and level N. Let g be a one-dimensional cusp for $\Gamma_0(N)$ of the form $\jmath(g_1(\gamma, x)^{-1})$. Then*

$$E_g(Z) = \sum_{\xi \in (gQ(\mathbb{Q})g^{-1} \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_1(g^{-1}\xi\langle Z\rangle^*) \det(j(g^{-1}\xi, Z))^{-k},$$

   defines a Klingen Eisenstein series of level N with respect to the Siegel congruence subgroup $\Gamma_0(N)$.

2. *Let h be a one-dimensional cusp for $\Gamma_0(N)$ of the form $\jmath(g_3(\gamma, \delta, y)^{-1})$. Let $f_2 \in S_k(\Gamma_{\jmath(h)})$ with even weight k such that $k \geq 6$. Then*

$$E_h(Z) = \sum_{\xi \in (hQ(\mathbb{Q})h^{-1} \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_2(h^{-1}\xi\langle Z\rangle^*) \det(j(h^{-1}\xi, Z))^{-k},$$

   defines a Klingen Eisenstein series of level N with respect to the Siegel congruence subgroup $\Gamma_0(N)$.

As g and h run through all one-dimensional cusps of the form $\jmath(g_1(\gamma, x)^{-1})$ and $\jmath(g_3(\gamma, \delta, y)^{-1})$ respectively, and for some fixed g and h, as f_1 and f_2 run through a basis of $S_k(\Gamma_0(N))$ and $S_k(\Gamma_h)$ respectively, the Klingen Eisenstein series thus obtained are linearly independent.

Proof. Since $k \geq 6$ and even the Klingen Eisenstein series defined in the statement of the Corollary have nice convergence properties. Let α and β be two one-dimensional cusps for $\Gamma_0(N)$. Let

$$E_\alpha(z) = \sum_{\xi \in (\alpha Q(\mathbb{Q})\alpha^{-1} \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_\alpha(\alpha^{-1}\xi\langle Z\rangle^*) \det(j(\alpha^{-1}\xi, Z))^{-k},$$

be a Klingen Eisenstein series associated to α. We have, $(E_\alpha|_k\beta)(Z)$

$$\begin{aligned} &= \sum_{\xi \in (\alpha Q(\mathbb{Q})\alpha^{-1} \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_\alpha(\alpha^{-1}\xi\langle\beta\langle Z\rangle\rangle^*) \det(j(\alpha^{-1}\xi, \beta\langle Z\rangle))^{-k} \det(j(\beta, Z))^{-k} \\ &= \sum_{\xi \in (\alpha Q(\mathbb{Q})\alpha^{-1} \cap \Gamma_0(N)) \backslash \Gamma_0(N)} f_\alpha(\alpha^{-1}\xi\beta\langle Z\rangle^*) \det(j(\alpha^{-1}\xi\beta, Z))^{-k}. \end{aligned}$$

Next consider $\Phi(E_\alpha|_k\beta)(z) = \lim_{\lambda\to\infty}(E_\alpha|_k\beta)(\left[\begin{smallmatrix} z & \\ & i\lambda \end{smallmatrix}\right])$ where Φ is the Siegel Φ operator defined earlier. The limit can be evaluated term by term because of nice convergence properties of the Eisenstein series. It follows from the proof of Prop. 5, Chap. 5, [14], that on taking limit only surviving terms are those with $\alpha^{-1}\xi\beta \in Q(\mathbb{Q})$ with $\xi \in \Gamma_0(N)$. If α and β are inequivalent cusps then clearly no term survives and $\Phi(E_\alpha|_k\beta)(z) = 0$. Whereas, we see that $\Phi(E_\alpha|_k\alpha)(z) = f_\alpha(z)$. We have shown that each Eisenstein series is supported on a unique one-dimensional cusp. Further for a fixed one-dimensional cusp all the associated Klingen Eisenstein series are clearly linearly independent. The result is now evident. $\square$

9.4 A co-dimension formula for cusp forms

The number of zero-dimensional cusps $C_0(p^n)$ for odd prime p was calculated by Markus Klein in his thesis (cf. Korollar 2.28 [12]).

$$C_0(p^n) = 2n + 1 + 2\left(\sum_{j=1}^{n-1} \phi(p^{\min(j,n-j)}) + \sum_{j=1}^{n-2}\sum_{i=j+1}^{n-1} \phi(p^{\min(j,n-i)})\right). \tag{9.1}$$

It is the same as

$$C_0(p^n) = \begin{cases} 3 & \text{if } n = 1, \\ 2p+3 & \text{if } n = 2, \\ -2n-1+2p^{\frac{n}{2}} + 8\frac{p^{\frac{n}{2}}-1}{p-1} & \text{if } n \geq 4 \text{ is even}, \\ -2n-1+6p^{\frac{n-1}{2}} + 8\frac{p^{\frac{n-1}{2}}-1}{p-1} & \text{if } n \geq 3 \text{ is odd}. \end{cases} \tag{9.2}$$

The above formula remains valid if $p = 2$ and $n = 1$. The above result also remains true for $p = 2$ and $n = 2$ as calculated by Tsushima (cf. [38]). Hence, assume $8 \nmid N$ and if $N = \prod_{i=1}^{m} p_i^{n_i}$ then following an argument similar to the one given in the proof of Lemma 8.2.1 we obtain

$$C_0(N) = \prod_{i=1}^{m} C_0(p_i^{n_i}). \tag{9.3}$$

Finally, by using Satake's theorem (cf. [34]) and the formula for $C_0(N)$ and $C_1(N)$ described earlier we obtain the following result.

Theorem 9.4.1. *Let $N \geq 1$, $8 \nmid N$ and $k \geq 6$, even, then*

$$\dim M_k(\Gamma_0(N)) - \dim S_k(\Gamma_0(N)) = C_0(N) + \left(\sum_{\gamma|N} \phi(\gcd(\gamma, \frac{N}{\gamma}))\right) \dim S_k(\Gamma_0^2(N))$$
$$+ \sum_{1<\delta<\gamma,\, \gamma|N,\, \delta|\gamma,\, \gamma|\delta^2} {\sum}' \dim S_k(\Gamma_g), \tag{9.4}$$

where $C_0(N)$ *is given by* (9.3) *if* $N > 1$, $C_0(1) = 1$, ϕ *denotes Euler's totient function, and for a fixed* γ *and* δ *the summation* $\sum'$ *is carried out such that* g *runs through every one-dimensional cusps of the form* $\jmath(g_3(\gamma, \delta, y))$, *with* y *taking all possible values as in Theorem 8.2.2, and* Γ_g *denotes* $\omega_1(g^{-1}\Gamma_0(\mathrm{N})g \cap Q(\mathbb{Q}))$.

Some remarks.

(i) We note that Markus Klein did not consider the case $4|N$ for calculating the number of zero-dimensional cusps in his thesis. Tsushima provided the result for $N = 4$. Since we refer to their results for the number of zero-dimensional cusps we have this restriction in our theorem. We hope to return to this case in the future.

(ii) The above result in the special case of square-free N reduces to the dimension formula given in [3] for even $k \geq 6$. [3] also treats the case $k = 4$ for square-free N.

(iii) We remind the reader that by $\jmath$ we denote the map that interchanges the first two rows and the first two columns of any matrix.

Proof. We consider the map $\tilde{\Phi} : M_k(\Gamma_0(N)) \to M_k(\mathrm{Bd}(N))$ by $F \to (\Phi(F|_k g_i))_{1 \leq i \leq l}$ which was described earlier in Sect. 9.2. The kernel of this map is the space of cusp forms $S_k(\Gamma_0(N))$. By Satake's theorem (cf. [34]), the map $\tilde{\Phi}$ is surjective. It follows that the co-dimension of the space of cusp forms is $\dim M_k(\mathrm{Bd}(N))$. We recall that by definition $f \in M_k(\mathrm{Bd}(N))$ means: f is a modular form of weight k on the boundary components of $S(N)$ such that f takes the same value on each intersection point of the boundary components. If $f \in S_k(\Gamma_i)$ on a boundary component $\Gamma_i \backslash \mathbb{H}_1$ corresponding to a one-dimensional cusp, say g_i, then f vanishes at every cusp of g_i and in particular f takes the same value zero at every intersection point of the boundary components. Hence $f \in M_k(\mathrm{Bd}(N))$. Since $k > 4$ and even, there exists a basis of (elliptic modular) Eisenstein series that is supported at a single zero-dimensional cusp. We denote the space spanned by such Eisenstein series as $E_k(\mathrm{Bd}(N))$. It is clear that $\dim E_k(\mathrm{Bd}(N)) = C_0(N)$, and if $f \in E_k(\mathrm{Bd}(N))$ then $f \in M_k(\mathrm{Bd}(N))$. Let C_0 and C_1 denote the set of zero and one-dimensional cusps of $\mathrm{Bd}(N)$ respectively. We have the following short exact sequence

$$0 \longrightarrow \oplus' S_k(\Gamma_g) \xrightarrow{\theta_1} M_k(\mathrm{Bd}(N)) \xrightarrow{\theta_2} \mathbb{C}^{C_0(N)} \xrightarrow{\sim} E_k(\mathrm{Bd}(N)) \longrightarrow 0.$$

Here $\oplus'$ denotes the direct sum as g runs through the set of one-dimensional cusps C_1 and the maps θ_1 and θ_2 are described in the following. First we note that any cusp form $f_g \in S_k(\Gamma_g)$ can be extended to the whole of $\mathrm{Bd}(N)$ by defining it to be zero on all the one-dimensional cusps other than g (i.e., f_g is trivial on the modular curve $\Gamma_h \backslash \mathbb{H}_1$ associated to any one-dimensional cusp h with $h \neq g$). We denote this extension also by the same symbol f_g. The map θ_1 is then the following map,

$$\theta_1((f_g)_{g \in C_1}) = \sum_{g \in C_1} f_g,$$

with f_g appearing on the right side of the equality being the extension of $f_g \in S_k(\Gamma_g)$ to the whole of $\mathrm{Bd}(N)$. Next the map θ_2 is defined by evaluating $f \in M_k(\mathrm{Bd}(N))$ at zero-dimensional cusps as follows

$$\theta_2(f) = (\lim_{z \to \infty} (f|_k h)(z))_{h \in C_0},$$

with C_0 being the set of zero-dimensional cusps of $\mathrm{Bd}(N)$ and $z \in \mathbb{H}_1$. It is clear that the image $\theta_2(M_k(\mathrm{Bd}(N)))$ is $\mathbb{C}^{C_0(N)}$, which is isomorphic to $E_k(\mathrm{Bd}(N))$. Hence we have $\dim M_k(\mathrm{Bd}(N)) = \dim E_k(\mathrm{Bd}(N)) + \dim S_k(\Gamma_g)$.

We note that the first terms in the summation formula (9.4) is $\dim E_k(\mathrm{Bd}(N)) = C_0(N)$ and it counts the Siegel Eisenstein series associated to each zero-dimensional cusp.

Next we show that the last two terms in the summation formula (9.4) add up to $\dim \oplus' S_k(\Gamma_g)$. It is easy to check that for any representatives g_1 of the form $\jmath(g_1(\gamma, x))$ with $g_1(\gamma, x)$ defined as in Theorem 8.2.2, we have $\omega_1(g_1^{-1}\Gamma_0(\mathrm{N})g_1 \cap Q(\mathbb{Q})) = \Gamma_0^2(N)$ and similarly for any representatives g_3 of the form $\jmath(g_3(\gamma, \delta, y))$ a simple calculation shows that $\Gamma_{g_3} = \omega_1(g_3^{-1}\Gamma_0(\mathrm{N})g_3 \cap Q(\mathbb{Q})) \subset \Gamma_0^2(\delta)$. It follows that each one-dimensional cusp of the form $\jmath(g_1(\gamma, x))$ contributes $\dim S_k(\Gamma_0^2(N))$ linearly independent cusp forms and this accounts for the second term in the formula (9.4). For a fixed δ and γ such that $1 < \delta < \gamma$, $\gamma|N$, $\delta|\gamma$, $\gamma|\delta^2$ and for a fixed y such that $y \in (\mathbb{Z}/L\mathbb{Z})^\times$ with $L = \gcd(\frac{\delta^2}{\gamma}, \frac{N}{\gamma})$, the one-dimensional cusp g of the form $\jmath(g_3(\gamma, \delta, y))$ contributes $\dim S_k(\Gamma_g)$ cusp forms. These contributions account for the last term in the summation formula (9.4). We remark that the last two terms in the summation formula (9.4) count the Klingen Eisenstein series associated to each one-dimensional cusp as defined in Theorem 9.3.1. $\square$

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