A NOTE ON DUAL MODULES AND THE TRANSPOSE

THOMAS MADSEN, ALAN ROCHE AND C. RYAN VINROOT

Abstract. It is a classical result in matrix algebra that a square matrix over a field can be conjugated to its transpose by a symmetric matrix. For F a non-Archimedean local field, Tupan used this to give an elementary proof that transpose inverse takes each irreducible smooth representation of $\mathrm{GL}_n(F)$ to its dual. We re-prove the matrix result and related observations using module-theoretic arguments. In addition, we write down a generalization that applies to central simple algebras with an involution of the first kind. We use this generalization to extend Tupan's method of argument to $\mathrm{GL}_n(D)$ for D a quaternion division algebra over F.

Introduction

Let F be a field and let a be a square matrix over F. Writing $\top$ for transpose, it is well known that there is an invertible matrix g over F such that $gag^{-1} = {}^{\top}a$ and ${}^{\top}g = g$ (see, for example, [2, 2.6] or [8]).

Our first object is to extend this classical matrix statement. We do so by replacing the pair $(\mathrm{M}_n(F), \top)$ by (A, θ) where A is a central simple algebra over F and θ is an involution on A of the first kind. By definition, the map θ is F-linear, reverses multiplication and satisfies ${}^{\theta}({}^{\theta}a) = a$ for all $a \in A$. For $\bar{F}$ an algebraic closure of F, we have $\bar{A} = A \otimes_F \bar{F} \cong \mathrm{M}_n(\bar{F})$ for $n^2 = \dim_F A$. The extended map $\bar{\theta} = \theta \otimes 1_{\bar{F}}$ is then an involution of the first kind on $\bar{A}$. For any $b \in \mathrm{GL}_n(\bar{F})$, we write $\mathrm{Int}\, b$ for the inner automorphism of $\mathrm{M}_n(\bar{F})$ given by conjugation by b. By a standard argument, any isomorphism $\bar{A} \cong \mathrm{M}_n(\bar{F})$ takes $\bar{\theta}$ to a composition $\mathrm{Int}\, b \circ \top$ where ${}^{\top}b = \varepsilon\, b$ for $\varepsilon = \pm 1$. The sign ε is independent of the choice of b and the choice of isomorphism $\bar{A} \cong \mathrm{M}_n(\bar{F})$. Accordingly, we write $\varepsilon = \varepsilon(\theta)$. Our extension of the classical matrix result is as follows:

$$(\star) \qquad \text{for any } a \in A, \text{ there is a } g \in A^{\times} \text{ such that } gag^{-1} = {}^{\theta}a \text{ and } {}^{\theta}g = \varepsilon(\theta)\, g.$$

Suppose now that F is a non-Archimedean local field. Tupan used the classical matrix result and some p-adic topology to give an elementary proof that transpose inverse takes each irreducible smooth representation of $\mathrm{GL}_n(F)$ to its dual. This was first established by Gelfand and Kazhdan by a geometric method [1]. Raghuram extended Gelfand-Kazhdan's method to the group $\mathrm{GL}_n(D)$ where D is a quaternion division algebra over F [5]. Using $(\star)$, it is a simple matter to extend Tupan's arguments to $\mathrm{GL}_n(D)$. We record the details in §2 below.

In the paper's final section, we re-prove the classical matrix statement and related observations from [8]. In place of matrix computations, we use some standard facts about finitely generated torsion modules over PIDs. While we certainly do not match the brevity or efficiency of the arguments in [8], there may be some merit in recording our conceptual approach.

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1. Conjugacy and Involutions

Let F be a field and let A be a central simple F-algebra. Thus the F-algebra A admits no proper nonzero two-sided ideals and has center F. For us also, A always has finite dimension as a vector space over F. Let θ be an involution on A. That is,

a) $\theta : A \to A$ is F-linear,
b) ${}^{\theta}(ab) = {}^{\theta}b\,{}^{\theta}a$ for all $a, b \in A$,
c) $\theta \circ \theta = 1_A$, the identity map on A.

In the literature, such maps are called *involutions of the first kind* in contrast to *involutions of the second kind* which satisfy only b) and c). Since we make no use here of involutions of the second kind, we use the term 'involution' from now on in place of the more cumbersome 'involution of the first kind.'

Attached to θ is a sign $\varepsilon(\theta) = \pm 1 \in F$ as discussed in the next two subsections. In the final subsection, we prove the conjugacy statement ($\star$) from the introduction.

Remark. There is a natural dichotomy – orthogonal versus symplectic – for involutions θ as above (see [3, 2.1] and the surrounding discussion). For $\operatorname{char} F \neq 2$, we have $\varepsilon(\theta) = 1$ (resp. -1) if θ is orthogonal (resp. symplectic). In the case $\operatorname{char} F = 2$, the dichotomy is irrelevant to our purposes.

1.1. **The split case.** We look first at the case of a matrix algebra $A = \mathrm{M}_n(F)$. As above, we write ${}^{\top}a$ for the transpose of any $a \in A$. For any involution θ on A, the composition $\theta \circ \top$ is an F-algebra automorphism of A. As such automorphisms are inner, there is a $b \in A^{\times}$ which is unique up to multiplication by an element of $F^{\times}$ such that

$${}^{\theta}({}^{\top}a) = bab^{-1}, \quad \forall\, a \in A.$$

Equivalently,

$$(1.1.1) \qquad {}^{\theta}a = b({}^{\top}a)b^{-1}, \quad \forall\, a \in A.$$

Using $\theta^2 = 1_A$, it follows that

$$a = b({}^{\top}b^{-1})\, a\, ({}^{\top}b)b^{-1}, \quad \forall\, a \in A,$$

and so $({}^{\top}b)b^{-1}$ is a scalar matrix. That is,

$${}^{\top}b = \varepsilon\, b, \quad \text{for some } \varepsilon \in F^{\times}.$$

Taking the transpose of each side, we obtain $b = \varepsilon^2\, b$. Thus $\varepsilon = \pm 1$, so the matrix b is symmetric or skew-symmetric. We put $\varepsilon(\theta) = \varepsilon$.

The sign $\varepsilon(\theta)$ can be expressed in terms of the eigenspaces of θ on A. We write $A^{\theta,+}$ and $A^{\theta,-}$ for the $+1$ and -1 eigenspaces of θ (resp.), so that $A = A^{\theta,+} \oplus A^{\theta,-}$ if $\operatorname{char} F \neq 2$. Using (1.1.1), one checks readily that

$${}^{\theta}a = \pm a \iff {}^{\top}(ab) = \pm\varepsilon\, ab.$$

For $\operatorname{char} F \neq 2$, it follows that $\varepsilon = \varepsilon(\theta)$ can be characterized by

$$(1.1.2) \qquad \dim_F A^{\theta,+} = \begin{cases} \dfrac{n(n+1)}{2} & \varepsilon = 1 \\ \dfrac{n(n-1)}{2} & \varepsilon = -1. \end{cases}$$

The formula still holds when $\operatorname{char} F = 2$ in the sense that the F-subspace of θ-fixed vectors also has dimension $\dfrac{n(n+1)}{2}$ in this case.

1.2. **The general case.** We return to the general setting. Thus A is a central simple F-algebra and θ is an involution on A.

As in the introduction, we write $\bar{F}$ for an algebraic closure of F and set $\bar{A} = A \otimes_F \bar{F}$. Then $\bar{A}$ is a central simple $\bar{F}$-algebra and hence $\bar{A} \cong \mathrm{M}_n(\bar{F})$ where $n^2 = \dim_F A$. The map θ extends to an involution $\bar{\theta}$ on $\bar{A}$ and so there is a sign $\varepsilon(\bar{\theta})$ as in §1.1. More pedantically, we set

$$\varepsilon(\bar{\theta}) = \varepsilon(\alpha\bar{\theta}\alpha^{-1})$$

for any $\bar{F}$-algebra isomorphism $\alpha : \bar{A} \to \mathrm{M}_n(\bar{F})$. By definition,

$$\varepsilon(\theta) = \varepsilon(\bar{\theta}).$$

Again, we write $A^{\theta,+}$ and $A^{\theta,-}$ for the $+1$ and -1 eigenspaces (resp.) of $\theta : A \to A$ and use a parallel notation for $\bar{A}$. The maps

$$a \otimes \lambda \mapsto \lambda a : A^{\theta,\pm} \otimes_F \bar{F} \xrightarrow{\simeq} \bar{A}^{\bar{\theta},\pm}$$

are then isomorphisms of $\bar{F}$-vector spaces. In particular, $\dim_F A^{\theta,+} = \dim_{\bar{F}} \bar{A}^{\bar{\theta},+}$. It follows that formula (1.1.2) also holds in this setting and characterizes the sign $\varepsilon(\theta)$. (Again we just use the first line ($\varepsilon = 1$) when $\operatorname{char} F = 2$.)

1.3. **Conjugacy result.** We can now state and prove our conjugacy result.

Theorem. *Let F be a field and let A be a central simple F-algebra with involution θ. We set $\varepsilon = \varepsilon(\theta)$. For any $a \in A$, there is a $g \in A^\times$ such that*

$$gag^{-1} = {}^\theta a \ \text{ and } \ {}^\theta g = \varepsilon\, g.$$

Proof. We look first at the split case $A = \mathrm{M}_n(F)$. By [2, page 76], there is an $h \in A^\times$ such that

$$hah^{-1} = {}^\top a \ \text{ and } \ {}^\top h = h.$$

With b as in (1.1.1), we have

$$\begin{aligned} (bh)a(bh)^{-1} &= b(hah^{-1})b^{-1} \\ &= b({}^\top a)b^{-1} \\ &= {}^\theta a. \end{aligned}$$

Moreover,

$$\begin{aligned} {}^\theta(bh) &= b\,{}^\top h\,{}^\top b\, b^{-1} \\ &= \varepsilon\, bh \qquad (\text{using } {}^\top h = h \text{ and } {}^\top b = \varepsilon\, b). \end{aligned}$$

This establishes the result in the split case.

Suppose now that A is non-split. In particular, the field F must be infinite. We use the notation introduced in §1.2. Thus $\bar{A} = A \otimes_F \bar{F} \cong \mathrm{M}_n(\bar{F})$ for $n^2 = \dim_F A$. We fix $a \in A$. Since the result holds in the split case, there is a $g \in \bar{A}^\times$ such that

$$gag^{-1} = {}^\theta a \ \text{ and } \ {}^{\bar{\theta}} g = \varepsilon\, g. \tag{1.3.1}$$

Write $\mathcal{S}$ for the set of x in A such that

$$xa = {}^{\theta}xa,\ {}^{\theta}x = \varepsilon\, x.$$

Then $\mathcal{S}$ is an F-subspace of A and $\bar{\mathcal{S}} = \mathcal{S} \otimes_F \bar{F}$ consists of all $x \in \bar{A}$ such that

$$xa = {}^{\bar{\theta}}xa,\ {}^{\bar{\theta}}x = \varepsilon\, x.$$

Note that $\bar{\mathcal{S}}$ contains the invertible element g of (1.3.1). To complete the proof, we show that $\mathcal{S}$ contains an invertible element. We do so by borrowing an argument from Raghuram (see the proof of [5, Lemma 3.1]) and Tupan (see the proof of [9, Lemma 2]).

We write $\mathrm{N}_A : A \to F$ and $\mathrm{N}_{\bar{A}} : \bar{A} \to \bar{F}$ for the reduced norm maps on A and $\bar{A}$ (resp.). We have

$$\mathrm{N}_{\bar{A}}(a \otimes 1) = \mathrm{N}_A(a), \quad a \in A. \tag{1.3.2}$$

Let $s_1, \ldots, s_m$ be a basis of $\mathcal{S}$ and consider the polynomial $f \in F[X_1, \ldots, X_m]$ such that

$$f(\lambda_1, \ldots, \lambda_m) = \mathrm{N}_A(\lambda_1 s_1 + \cdots + \lambda_m s_m)$$

for $\lambda_1, \ldots, \lambda_m \in F$. Given $\bar{\lambda}_1, \ldots, \bar{\lambda}_m \in \bar{F}$, it follows from (1.3.2) that

$$\mathrm{N}_{\bar{A}}(s_1 \otimes \bar{\lambda}_1 + \cdots + s_m \otimes \bar{\lambda}_m) = f(\bar{\lambda}_1, \ldots, \bar{\lambda}_m).$$

By (1.3.1), $\bar{\mathcal{S}}$ contains the invertible element g, so that $\mathrm{N}_{\bar{A}}(g) \neq 0$. In particular, f is not the zero polynomial. As F is infinite, it follows that there exist $\lambda_1, \ldots, \lambda_m \in F$ such that $f(\lambda_1, \ldots, \lambda_m) \neq 0$. That is,

$$\mathrm{N}_A(g') \neq 0 \ \text{ for } g' = \lambda_1 s_1 + \cdots + \lambda_m s_m \in \mathcal{S}.$$

Thus g' is invertible, so we've completed the proof. □

2. An Application

Let F be a non-Archimedean local field and let D be a quaternion division algebra over F. We write γ for the canonical conjugation on D. It is the unique symplectic involution on D ([3, 2.21]). In particular, $\varepsilon(\gamma) = -1$. Given $a = (a_{ij}) \in \mathrm{M}_n(D)$, we set ${}^{\gamma}a = ({}^{\gamma}a_{ij})$ and ${}^{\theta}a = {}^{\top\gamma}a$ (so that ${}^{\theta}a$ has ij entry ${}^{\gamma}a_{ji}$.) The resulting map θ defines an involution on $\mathrm{M}_n(D)$. We have $\varepsilon(\theta) = -1$ (by a direct calculation or by [3, 2.20]). Finally, we define an involutory automorphism ι of $\mathrm{GL}_n(D)$ by ${}^{\iota}g = {}^{\theta}g^{-1}$, $g \in \mathrm{GL}_n(D)$.

For π an irreducible smooth (complex) representation of G, we write $\pi^{\vee}$ for the smooth dual or contragredient of π. It is well known that $\pi \circ \iota \cong \pi^{\vee}$ for any such π. In the terminology of [7, 6], ι is a *dualizing involution.* This was proved by Muić and Savin in the case $\operatorname{char} F = 0$ [4] using character theory and by Raghuram in all characteristics [5]. Raghuram's proof is an adaptation of a geometric method used by Gelfand and Kazhdan to show that $a \mapsto {}^{\top}a^{-1}$ is a dualizing involution on $\mathrm{GL}_n(F)$ [1]. As noted in the introduction, Tupan found a completely elementary proof of Gelfand-Kazhdan's result using a) the classical observation that a square matrix over a field is conjugate to its transpose via a symmetric matrix and b) some p-adic topology [9]. Our object in this section is to show that Tupan's method carries over to the group $\mathrm{GL}_n(D)$ using Theorem 1.3 in place of a).

2.1. **Axiomatic version of Tupan's method.** It is convenient to use the axiomatic version of Tupan's method from [6]. Thus let G be the group of F-points of a reductive algebraic group over F and write $\mathfrak{g}$ for the Lie algebra of G. Let $\vartheta : G \to G$ be an involutary anti-isomorphism on G (induced by a corresponding map on the underlying algebraic group). We also write ϑ for the induced map on $\mathfrak{g}$. As usual, for $x \in G$, we write $\mathrm{Int}(x)$ for the automorphism of G given by conjugation by x and $\mathrm{Ad}(x)$ for the induced map on $\mathfrak{g}$.

Let $\mathfrak{o}_F$ denote the valuation ring of F and fix a uniformizer ϖ in F. Consider the following hypotheses.

(1) There is an $\mathfrak{o}_F$-lattice $\mathcal{L} \subset \mathfrak{g}$ and a map $c : \mathfrak{g}_1 \to G$ for a certain subset $\mathfrak{g}_1$ of $\mathfrak{g}$ such that the following hold.

- (a) ${}^{\vartheta}\mathfrak{g}_1 = \mathfrak{g}_1$ and $\vartheta \circ c = c \circ \vartheta$.
- (b) $\mathrm{Ad}(x)\mathfrak{g}_1 = \mathfrak{g}_1$ and $\mathrm{Int}(x)c(X) = c(\mathrm{Ad}(x)X)$ for all $x \in G$ and $X \in \mathfrak{g}_1$.
- (c) ${}^{\vartheta}\mathcal{L} = \mathcal{L}$ and $\varpi\mathcal{L} \subset \mathfrak{g}_1$.
- (d) For each $k \geq 1$, the restriction $c \mid \varpi^k\mathcal{L}$ is a homeomorphism onto a compact open subgroup of G. In particular, the family $\{c(\varpi^k\mathcal{L})\}_{k\geq 1}$ consists of compact open subgroups and forms a neighborhood basis of the identity in G.

(2) For each $a \in G$, there is a $g \in G$ with ${}^{\vartheta}g = g$ such that $gag^{-1} = {}^{\vartheta}a$.

Let ${}^{\iota}g = {}^{\vartheta}g^{-1}$, $g \in G$. Subject to these hypotheses, [6, Theorem 2.2] shows that the resulting map $\iota : G \to G$ is a dualizing involution.

2.2. **The case $G = \mathrm{GL}_n(D)$.** We apply the framework of §2.1 to $G = \mathrm{GL}_n(D)$. We have $\mathfrak{g} = \mathrm{M}_n(D)$ and $\mathrm{Ad}(x)X = xXx^{-1}$ for $x \in G$ and $X \in \mathfrak{g}$. We write $\mathrm{Nrd} : \mathrm{M}_n(D) \to F$ for the reduced norm map and $\mathfrak{o}_D$ for the the unique maximal $\mathfrak{o}_F$-order in D.

Suppose first that $\operatorname{char} F = 2$. We take $\mathfrak{g}_1 = \{X \in \mathrm{M}_n(D) : \mathrm{Nrd}(1 + X) \neq 0\}$ and define $c : \mathfrak{g}_1 \to G$ by $c(X) = 1 + X$. We set $\mathcal{L} = \mathrm{M}_n(\mathfrak{o}_D)$. With $\vartheta = \theta$, it is then immediate that (a)-(d) of (1) hold. Hypothesis (2) holds by Theorem 1.3. Thus ${}^{\iota}g = {}^{\theta}g^{-1}$, $g \in G$, defines a dualizing involution.

Suppose now that $\operatorname{char} F \neq 2$. Choose any $y \in D^{\times}$ with ${}^{\gamma}y = -y$. For simplicity, we also write y for the matrix yI_n in G. For $a \in \mathrm{M}_n(D)$, we set ${}^{\vartheta}a = y\,{}^{\theta}a\,y^{-1}$. Since ${}^{\gamma}y = -y$, the resulting map ϑ is an involution. By a direct calculation,

$$ {}^{\vartheta}a = a \iff {}^{\theta}(ay) = -ay. $$

It follows that $\varepsilon(\vartheta) = 1$. We set

$$ \mathcal{L} = \mathrm{M}_n(\mathfrak{o}_D) \cap y\,\mathrm{M}_n(\mathfrak{o}_D)\,y^{-1}. $$

Note that ${}^{\gamma}(y^2) = y^2$, so that $y^2 \in F$. Thus

$$ y\,\mathcal{L}\,y^{-1} = y\,\mathrm{M}_n(\mathfrak{o}_D)\,y^{-1} \cap y^2\,\mathrm{M}_n(\mathfrak{o}_D)\,y^{-2} = \mathcal{L}. \tag{2.2.1} $$

Similarly,

$$ \begin{aligned} {}^{\theta}\mathcal{L} &= {}^{\theta}\mathrm{M}_n(\mathfrak{o}_D) \cap {}^{\theta}y^{-1}\,{}^{\theta}\mathrm{M}_n(\mathfrak{o}_D)\,{}^{\theta}y \\ &= \mathrm{M}_n(\mathfrak{o}_D) \cap y^{-1}\,\mathrm{M}_n(\mathfrak{o}_D)\,y \quad (\text{using } {}^{\theta}y = -y) \end{aligned} \tag{2.2.2} $$

$$\begin{aligned} &= \mathrm{M}_n(\mathfrak{o}_D) \cap y\,\mathrm{M}_n(\mathfrak{o}_D)\,y^{-1} \quad (\text{as } y^2 \in F) \\ &= \mathcal{L}. \end{aligned}$$

By (2.2.1) and (2.2.2), we have ${}^{\vartheta}\mathcal{L} = \mathcal{L}$. Again we take $\mathfrak{g}_1 = \{X \in \mathrm{M}_n(D) : \mathrm{Nrd}(1+X) \neq 0\}$ and define $c : \mathfrak{g}_1 \to G$ by $c(X) = 1 + X$. Then (a)-(d) of (1) hold. Hypothesis (2) holds once more by Theorem 1.3. Hence $g \mapsto {}^{\vartheta}g^{-1}$ defines a dualizing involution of G. As ϑ and θ differ by an inner automorphism, it follows that ${}^{\iota}g = {}^{\theta}g^{-1}$, $g \in G$, is also a dualizing involution.

3. Revisiting the matrix algebra case

Let n be a positive integer and let $a \in \mathrm{M}_n(F)$. We set $V = F^n$, viewed as a set of column vectors, and write V_a for the $F[X]$-module structure on V given by $f(X)v = f(a)v$ for $f(X) \in F[X]$ and $v \in V$. We wish to prove the following.

Theorem. *Given $a \in \mathrm{M}_n(F)$, there is a $g \in \mathrm{GL}_n(F)$ such that*

$$(1)\ gag^{-1} = {}^{\top}a \ \text{ and } \ (2)\ {}^{\top}g = g.$$

Moreover, if V_a is a cyclic module then any $g \in \mathrm{GL}_n(F)$ that satisfies (1) also satisfies (2). If V_a is non-cyclic then there is a $g \in \mathrm{GL}_n(F)$ that satisfies (1) but not (2). In other words, every $g \in \mathrm{GL}_n(F)$ that satisfies (1) also satisfies (2) if and only if the minimal and characteristic polynomials of a coincide.

The result is not new – see [8]. Our goal is to provide a conceptual proof that relies principally on standard facts about finitely generated torsion modules over PIDs and makes minimal use of special matrix calculations.

Proof. The argument is spread over the next several subsections.

3.1. **Reduction to cyclic case.** There is an $h \in \mathrm{GL}_n(F)$ such that

$$hah^{-1} = \begin{bmatrix} a_1 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & 0 & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_r \end{bmatrix}$$

with $a_i \in \mathrm{M}_{n_i}(F)$ and each $(F^{n_i})_{a_i}$ cyclic $(i = 1, \ldots, r)$. Suppose $b_i \in \mathrm{GL}_{n_i}(F)$ satisfies

$$b_i a_i b_i^{-1} = {}^{\top}a_i \ \text{ and } \ {}^{\top}b_i = b_i \qquad (i = 1, \ldots, r).$$

With

$$b = \begin{bmatrix} b_1 & 0 & \cdots & \cdots & 0 \\ 0 & b_2 & 0 & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & b_r \end{bmatrix},$$

we then have ${}^\top b = b$ and

$$b(hah^{-1})b^{-1} = {}^\top(hah^{-1}).$$

Rearranging gives

$$({}^\top hbh)a({}^\top hbh)^{-1} = {}^\top a$$

with ${}^\top({}^\top hbh) = {}^\top hbh$.

3.2. **Generalities.** We recall some generalities about $F[X]$-modules that we will apply eventually to V_a.

For any $F[X]$-module M, we set $M^\vee = \mathrm{Hom}_F(M, F)$. We write $\langle\ ,\ \rangle$ for the canonical (evaluation) pairing between M and $M^\vee$:

$$\langle m, m^\vee\rangle = m^\vee(m), \qquad m \in M,\ m^\vee \in M^\vee.$$

The space $M^\vee$ is an $F[X]$-module via

$$\langle m, f(X)m^\vee\rangle = \langle f(X)m, m^\vee\rangle, \qquad m \in M,\ m^\vee \in M^\vee.$$

For $m \in M$, let $\epsilon(m) = \langle m, -\rangle$, so that $\epsilon(m) \in M^{\vee\vee}$. The resulting map

$$\epsilon = \epsilon_M : M \to M^{\vee\vee} \tag{3.2.1}$$

is an $F[X]$-module homomorphism. Thus, if M has finite dimension over F, then (3.2.1) is an isomorphism of $F[X]$-modules.

Suppose now that M is a torsion $F[X]$-module that is also finite dimensional as an F-vector space. Let $\mathcal{P}$ denote the set of monic irreducible polynomials in $F[X]$. For $p \in \mathcal{P}$, we write $M[p]$ for the p-primary component of M:

$$M[p] = \{m \in M : p^e m = 0 \text{ for some positive integer } e\}.$$

Then

$$M = \bigoplus_{p\in\mathcal{P}} M[p]. \tag{3.2.2}$$

Moreover, for any $p \in \mathcal{P}$, there is a canonical isomorphism

$$M^\vee[p] \;\cong\; M[p]^\vee,$$

so that

$$M^\vee \cong \bigoplus_{p\in\mathcal{P}} M[p]^\vee,$$

which also follows directly from (3.2.2). By the Chinese Remainder Theorem, M is cyclic if and only if each of its p-primary components $M[p]$ is cyclic. Further, for each $p \in \mathcal{P}$, $M[p]$ is cyclic if and only if $M[p]$ is indecomposable. Observe next that $M[p]$ is indecomposable if and only if $M[p]^\vee$ is indecomposable (for any $p \in \mathcal{P}$). Indeed, if $M[p]$ splits as a non-trivial direct sum, then the same holds for $M[p]^\vee$. For the other direction, note that

$$M[p]^{\vee\vee} \cong M^{\vee\vee}[p] \cong M[p].$$

Thus if $M[p]^\vee$ is a non-trivial direct sum then $M[p]$ splits in the same way. It follows that M is cyclic if and only if $M^\vee$ is cyclic.

Write $\operatorname{ann} M$ for the annihilator of the $F[X]$-module M:

$$\operatorname{ann} M = \{f(X) \in F[X] : f(X)m = 0 \text{ for all } m \in M\}.$$

Note that

$$\operatorname{ann} M = \operatorname{ann} M^\vee. \tag{3.2.3}$$

Indeed,

$$\begin{aligned} f(X) \in \operatorname{ann} M &\iff \langle f(X)m, m^\vee\rangle = 0, \ \ \forall\, m \in M,\, m^\vee \in M^\vee \\ &\iff \langle m, f(X)m^\vee\rangle = 0, \ \ \forall\, m \in M,\, m^\vee \in M^\vee \\ &\iff f(X) \in \operatorname{ann} M^\vee. \end{aligned}$$

Thus, for M cyclic,

$$\begin{aligned} M &\cong F[X]/\operatorname{ann} M \\ &= F[X]/\operatorname{ann} M^\vee \quad (\text{by } (3.2.3)) \\ &\cong M^\vee. \end{aligned}$$

In general, M is a direct sum of cyclic submodules. Since each of these cyclic summands is self-dual, we see again that $M \cong M^\vee$. Taking $M = V_a$, we have an isomorphism of $F[X]$-modules

$$\eta : V_a \xrightarrow{\simeq} V_a^\vee. \tag{3.2.4}$$

Let $(\ ,\ )$ denote the usual dot product on V, that is,

$$(v, w) = {}^\top v w, \qquad v, w \in V.$$

Then, for any $b \in \mathrm{M}_n(F)$ and $v, w \in V$,

$$(bv, w) = (v, {}^\top b w).$$

In particular,

$$({}^\top a v, w) = (v, aw), \qquad v, w \in V.$$

Thus, if we set $\gamma(v) = (v, -)$ for $v \in V$, then

$$\gamma : V_{{}^\top a} \xrightarrow{\simeq} V_a^\vee$$

is an isomorphism of $F[X]$-modules. Hence

$$\gamma^{-1} \circ \eta : V_a \xrightarrow{\simeq} V_{{}^\top a}$$

is an isomorphism of $F[X]$-modules. This means there is a $g \in \mathrm{GL}_n(F)$ such that

$$g a g^{-1} = {}^\top a. \tag{3.2.5}$$

3.3. **The cyclic case.** We write ${}^{\top}\eta : V_a^{\vee\vee} \to V_a^{\vee}$ for the dual isomorphism to (3.2.4) and set $\eta^{\vee} = {}^{\top}\eta \circ \epsilon$ with ϵ as in (3.2.1) (for $M = V_a$). Thus $\eta^{\vee} : V_a \to V_a^{\vee}$ is again an isomorphism of $F[X]$-modules. Unwinding the definitions, one checks that it is characterized by the identity

$$\langle v, \eta^{\vee}(w)\rangle = \langle w, \eta(v)\rangle, \qquad \forall\ v, w \in V. \tag{3.3.1}$$

We wish to show that if V_a is cyclic then any g that satisfies (3.2.5) is necessarily symmetric. The crux of our argument is the following.

Lemma. *Assume that V_a is cyclic. Then, with notation as above, $\eta^{\vee} = \eta$.*

Proof. Let v_1 be a generator of V_a. In (3.3.1), we can write $v = f(X)v_1$ and $w = g(X)v_1$ for suitable $f(X)$, $g(X) \in F[X]$. Then

$$\begin{aligned}\langle w, \eta(v)\rangle &= \langle g(X)v_1, \eta(f(X)v_1)\rangle \\ &= \langle f(X)g(X)v_1, \eta(v_1)\rangle.\end{aligned}$$

In the same way,

$$\langle v, \eta^{\vee}(w)\rangle = \langle f(X)g(X)v_1, \eta^{\vee}(v_1)\rangle.$$

Hence

$$\langle g(X)v_1, \eta(f(X)v_1)\rangle = \langle g(X)v_1, \eta^{\vee}(f(X)v_1)\rangle.$$

Thus

$$\langle w, \eta(v)\rangle = \langle w, \eta^{\vee}(v)\rangle, \qquad \forall\ v, w \in V,$$

and so $\eta = \eta^{\vee}$ as claimed. $\square$

The matrix g of (3.2.5) satisfies

$$gv = \gamma^{-1}(\eta v),$$

or $\gamma(gv) = \eta(v)$, for $v \in V$. Hence

$$(gv, w) = \langle w, \eta(v)\rangle, \qquad \forall\ v, w \in V.$$

Therefore

$$\begin{aligned}(v, gw) &= (gw, v) \\ &= \langle v, \eta(w)\rangle, \qquad \forall\ v, w \in V.\end{aligned}$$

Thus, for all $v, w \in V$,

$$(gv, w) = (v, gw) \iff \langle w, \eta(v)\rangle = \langle v, \eta(w)\rangle.$$

Using (3.3.1), it follows that

$${}^{\top}g = g \iff \eta^{\vee} = \eta. \tag{3.3.2}$$

Hence, by Lemma 3.3, ${}^{\top}g = g$ whenever V_a is cyclic.

3.4. **The non-cyclic case.** Suppose now that V_a is non-cyclic. It remains to show that there is a $g \in \mathrm{GL}_n(F)$ such that $gag^{-1} = {}^\top a$ with ${}^\top g \neq g$. By (3.3.2), it suffices to show that $\eta^\vee \neq \eta$ with η as in (3.2.4).

As V_a is non-cyclic, we can write

$$V_a = V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

where a) each V_i is a cyclic submodule and b) $\mathrm{Hom}_{F[X]}(V_1, V_2^\vee) \neq \{0\}$. For example, as noted above, some p-primary component $V_a[p]$ must be non-cyclic, so this component splits as a non-trivial direct sum $V_1 \oplus V_2$. As $F[X]/(p)$ is the unique composition factor of V_1 and $V_2^\vee$, condition b) surely holds.

We fix a non-zero $F[X]$-module homomorphism $\eta_{12} : V_1 \to V_2^\vee$. Dualizing gives a non-zero $F[X]$-module map ${}^\top\eta_{12} : V_2^{\vee\vee} \to V_1^\vee$. Then $\eta_{12}^\vee = \eta_{12} \circ \epsilon_{V_2}$ is a non-zero $F[X]$-module map from V_2 to $V_1^\vee$. We also fix $F[X]$-module isomorphisms $\eta_i : V_i \to V_i^\vee$ (for $i = 1, \ldots, r$). Using block matrix notation, we set

$$\eta = \begin{bmatrix} \eta_1 & \eta_{12} & 0 & \cdots & 0 \\ 0 & \eta_2 & 0 & & \vdots \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \eta_r \end{bmatrix},$$

so that $\eta : V_a \to V_a^\vee$ is an isomorphism of $F[X]$-modules. We have

$$\eta^\vee = \begin{bmatrix} \eta_1 & 0 & 0 & \cdots & 0 \\ \eta_{12}^\vee & \eta_2 & 0 & & \vdots \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \eta_r \end{bmatrix}.$$

In particular, $\eta^\vee \neq \eta$. This completes the proof. $\square$

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Dept. of Mathematics and Statistics, Youngstown State University, Youngstown, OH 44555.
E-mail address: tlmadsen@ysu.edu

Dept. of Mathematics, University of Oklahoma, Norman, OK 73019-3103.
E-mail address: aroche@ou.edu

Dept. of Mathematics, College of William and Mary, P.O. Box 8795, Williamsburg, VA 23187-8795.
E-mail address: vinroot@math.wm.edu