MATH 3333

Midterm II

April 3, 2008

Name :

I.D. no. :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.
- Show as much work as possible. Answers without explanation will not receive any credit.
- If you preform any row or column operations in a problem, record them using standard notations.
- Best of Luck.

i) Let $V = R^4$ and W be the subset of R^4 consisting of all vectors of the form $\begin{bmatrix} a-b \\ a+c \\ c+b \\ a-c \end{bmatrix}$,

where a, b, c are real numbers.

a) (8 Points) Show that W is a vector subspace of R^4.

- Let $\mathbf{w}_1 = \begin{bmatrix} a_1 - b_1 \\ a_1 + c_1 \\ c_1 + b_1 \\ a_1 - c_1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} a_2 - b_2 \\ a_2 + c_2 \\ c_2 + b_2 \\ a_2 - c_2 \end{bmatrix}$ be two elements of W.

$$\mathbf{w}_1 \oplus \mathbf{w}_2 = \begin{bmatrix} (a_1 - b_1) + (a_2 - b_2) \\ (a_1 + c_1) + (a_2 + c_2) \\ (c_1 + b_1) + (c_2 + b_2) \\ (a_1 - c_1) + (a_2 - c_2) \end{bmatrix} = \begin{bmatrix} (a_1 + a_2) - (b_1 + b_2) \\ (a_1 + a_2) + (c_1 + c_2) \\ (c_1 + c_2) + (b_1 + b_2) \\ (a_1 + a_2) - (c_1 + c_2) \end{bmatrix}$$

  which is also in W. Hence W is closed under $\oplus$.

- Let r be a real number. Then

$$r \odot \mathbf{w}_1 = \begin{bmatrix} r(a_1 - b_1) \\ r(a_1 + c_1) \\ r(c_1 + b_1) \\ r(a_1 - c_1) \end{bmatrix} = \begin{bmatrix} (ra_1) - (rb_1) \\ (ra_1) + (rc_1) \\ (rc_1) + (rb_1) \\ (ra_1) - (rc_1) \end{bmatrix}$$

  which is also in W. Hence W is closed under $\odot$.

This shows that W is a vector subspace of R^4.

b) (12 Points) Find a basis for and dimension of W.

Note that for any vector $\mathbf{w}$ in W, we have $\mathbf{w} = \begin{bmatrix} a-b \\ a+c \\ c+b \\ a-c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} +$ $c \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$. Hence $W = \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Now, $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = 0 \rightarrow a_1 - a_2 = 0, a_1 + a_3 = 0, a_3 + a_2 = 0, a_1 - a_3 = 0 \Rightarrow a_1 = a_2 = a_3 = 0$. This means that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent. Hence, a basis for W is given by

$$\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \}$$

This implies that $\dim(W) = 3$.

ii) Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

a) (10 Points) Find $\mathrm{Adj}(A)$.

We have to calculate the cofactors.

$$A_{11} = (-1)^{1+1}\det(\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}) = 4, \qquad A_{12} = (-1)^{1+2}\det(\begin{bmatrix} 4 & 6 \\ 7 & 2 \end{bmatrix}) = 34,$$

$$A_{13} = (-1)^{1+3}\det(\begin{bmatrix} 4 & 5 \\ 7 & 1 \end{bmatrix}) = -31, \qquad A_{21} = (-1)^{2+1}\det(\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}) = 4$$

$$A_{22} = (-1)^{2+2}\det(\begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}) = -8, \qquad A_{23} = (-1)^{2+3}\det(\begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}) = -10$$

$$A_{31} = (-1)^{3+1}\det(\begin{bmatrix} -1 & 2 \\ 5 & 6 \end{bmatrix}) = -16, \qquad A_{32} = (-1)^{3+2}\det(\begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}) = -10,$$

$$A_{33} = (-1)^{3+3}\det(\begin{bmatrix} 3 & -1 \\ 4 & 5 \end{bmatrix}) = 19.$$

Hence the adjoint of A is given by

$$\mathrm{Adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 4 & 4 & -16 \\ 34 & -8 & -10 \\ -31 & -10 & 19 \end{bmatrix}$$

b) (10 Points) Find $\det(A)$ by expanding along the second column.

Expanding along the second column we get

$$\begin{aligned} \det(A) &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \\ &= (-1)(34) + (5)(-8) + (1)(-10) \\ &= -34 - 40 - 10 \\ &= -84 \end{aligned}$$

c) (10 Points) Find A^{-1}, if it exists. (Hint : If you want, you can use parts (a) and (b) above to avoid lengthy calculations)

From part (b) on the previous page $\det(A) = -84 \neq 0$. This implies that A^{-1} exists. We have

$$A^{-1} = \frac{1}{\det(A)}\text{Adj}(A) = \frac{-1}{84}\begin{bmatrix} 4 & 4 & -16 \\ 34 & -8 & -10 \\ -31 & -10 & 19 \end{bmatrix}.$$

d) (10 Points) Find all solutions to the linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Since A is invertible, the above linear system has only one solution given by

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{-1}{84}\begin{bmatrix} 4 & 4 & -16 \\ 34 & -8 & -10 \\ -31 & -10 & 19 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/7 \\ -2/7 \\ 1/7 \end{bmatrix}$$

iii) Let

$$A = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$$

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a) (15 Points) Find a basis for the Row space of A.

We know that the basis for the row space for the matrix A consists of the non-zero rows in the row echelon form of the matrix.

$$\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix} \xrightarrow[-2\mathbf{r}_1+\mathbf{r}_5\to\mathbf{r}_5]{-\mathbf{r}_1+\mathbf{r}_4\to\mathbf{r}_4} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & -4 & -1 & 0 \\ 0 & -1 & -2 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow[\mathbf{r}_2+\mathbf{r}_5\to\mathbf{r}_5]{2\mathbf{r}_2+\mathbf{r}_4\to\mathbf{r}_4} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix} \xrightarrow[\mathbf{r}_3+\mathbf{r}_5\to\mathbf{r}_5]{-\mathbf{r}_3+\mathbf{r}_4\to\mathbf{r}_4} \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, we get that basis for the row space of A is given by

$$\{\begin{bmatrix} 1 & 1 & 4 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \end{bmatrix}\}.$$

b) (5 Points) Find Rank(A) and Nullity(A).

We know that

$$\text{Rank}(A) = \text{RowRank}(A) = \dim(\text{Rowspaceof}A) = 3.$$

To get Nullity we use the following formula

$$\text{Rank}(A) + \text{Nullity}(A) = 5 \Rightarrow \text{Nullity}(A) = 5 - \text{Rank}(A) = 5 - 3 = 2.$$

iv) State whether the following statements are true or false. Explain your answers.

a) (6 Points) Let $V = R_2$ and W be the subset of R_2 consisting of all vectors of the form $\begin{bmatrix} x & 2x^2 \end{bmatrix}$, where x is any real number. Then W is a vector subspace of V.

FALSE

If $(x_1, 2x_1^2)$ and $(x_2, 2x_2^2)$ are two elements of W, then $(x_1, 2x_1^2) + (x_2, 2x_2^2) = (x_1 + x_2, 2(x_1^2 + x_2^2))$. This element does not belong to W because, in general, $(x_1 + x_2)^2 \neq x_1^2 + x_2^2$. Hence, W is **NOT** a vector subspace of R_2.

b) (8 Points) Let A be a $n \times n$ matrix such that $A^T A$ is non-singular. Then $\text{Rank}(A) = n$.

TRUE

Since $A^T A$ is non-singular, we have $\det(A^T A) \neq 0$. Since $\det(A^T A) = \det(A^T)\det(A)$, we get that $\det(A) \neq 0$. This means that A is also non-singular. We know that if A is a $n \times n$ matrix then A is non-singular if and only if $\text{Rank}(A) = n$. Hence, we can conclude that $\text{Rank}(A) = n$.

c) (6 Points) $\det\left(\begin{bmatrix} a+b & ab \\ 1 & a+b \end{bmatrix}\right) = (a^3 - b^3)/(a-b)$.

TRUE

$$\begin{aligned}\det\left(\begin{bmatrix} a+b & ab \\ 1 & a+b \end{bmatrix}\right) &= (a+b)(a+b) - ab = (a^2 + 2ab + b^2) - ab = a^2 + ab + b^2 \\ &= \frac{(a^2 + ab + b^2)(a-b)}{(a-b)} = \frac{a^3 - b^3}{a-b}.\end{aligned}$$