MATH 3333

Final

May 8, 2008

<u>Name</u> :

<u>I.D. no.</u> :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.
- Show as much work as possible. Answers without explanation will not receive any credit.
- If you preform any row or column operations in a problem, record them using standard notations.
- Best of Luck.

i) Let $L : M_{22} \to M_{22}$ be the function given by

$$L(A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A - A^T \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

a) (10 Points) Show that L is a linear transformation.

b) (10 Points) Which of the following matrices are in $\mathrm{Ker}(L)$? Explain.

$$A_1 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}.$$

c) (5 Points) Is L one-to-one ? Explain.

ii) (25 Points) Let S be the standard basis for R^3 and $T = \{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\}$ be another basis for R^3. Consider the linear transformation $L : R^3 \to R^3$ given by

$$L(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}) = \begin{bmatrix} u_1 + u_2 \\ u_2 + u_3 \\ u_3 + u_1 \end{bmatrix}$$

Find the matrix representing L with respect to the ordered basis S and T.

iii) Let

$$A = \begin{bmatrix} 0 & -4 & 2 \\ 2 & 6 & -2 \\ 0 & 0 & 2 \end{bmatrix}.$$

a) (20 Points) Find the characteristic polynomial, eigenvalues and eigenvectors of A.

b) (5 Points) State whether A is diagonalizable. Explain. (Do not calculate the matrix P and its inverse)

iv) (20 Points) Find a basis for R^3 that includes the vectors $\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \}$.

v) (20 Points) Let

$$A = \begin{bmatrix} 1 & -2 & -3 & 4 \\ 4 & -3 & -7 & 6 \\ 2 & 1 & -1 & -2 \end{bmatrix}$$

Find a basis for the null space of A.

vi) (20 Points) Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 2 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}$$

Find $\det(A)$.

vii) Let $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$.

a) (15 Points) Find P^{-1}

b) (10 Points) Let $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A = PDP^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find A^9.

(You may need $2^9 = 512$)

viii) State whether the following statements are True or False. Explain your answer.

a) (10 Points) Let A, B, C, D be 2×2 invertible matrices. Then

$$((AB^TC^{-1}D)^T)^{-1} = (A^T)^{-1}B^{-1}C^T(D^T)^{-1}$$

b) (10 Points) Let $A = \begin{bmatrix} \cos(20) & -\sin(20) \\ \sin(20) & \cos(20) \end{bmatrix}$. Then $A^{15}\mathbf{u} = \mathbf{u}$ for all $\mathbf{u}$ in R^2.

ix) (20 Points) A $n \times n$ matrix A is called an **orthogonal** matrix if it satisfies $A^T A = I_n$. State which of the following matrices are orthogonal.

$$A_1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$