Math 3333, Quiz IV, October 11, 2007

Name :

1. (5 Points) Show that if A is a $n \times n$ matrix then $\det(AA^T) \geq 0$.

$$\det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = \det(A)^2 \geq 0.$$

We have used the fact that $\det(A) = \det(A^T)$ and the square of any real number is non-negative.

2. (10 Points) Use Cramer's rule to solve the following linear system

$$2x_1 + 3x_2 + 4x_3 = 2,\ x_1 + 2x_2 + 4x_3 = 0,\ 4x_1 + 3x_2 + x_3 = 0.$$

We have $A\mathbf{x} = \mathbf{b}$ with $A = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 4 \\ 4 & 3 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

$$\det(A) = 5, \qquad \det(A_1) = \det(\begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & 4 \\ 0 & 3 & 1 \end{pmatrix}) = -20,$$

$$\det(A_2) = \det(\begin{pmatrix} 2 & 2 & 4 \\ 1 & 0 & 4 \\ 4 & 0 & 1 \end{pmatrix}) = 30, \qquad \det(A_3) = \det(\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 0 \\ 4 & 3 & 0 \end{pmatrix}) = -10.$$

By Cramer's rule, we have

$$x_1 = \frac{\det(A_1)}{\det(A)} = -4, \qquad x_2 = \frac{\det(A_2)}{\det(A)} = 6, \qquad x_3 = \frac{\det(A_3)}{\det(A)} = -2.$$

3. (5 Points) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in a vector space V. Show that if $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} \oplus \mathbf{w}$ then we must have $\mathbf{v} = \mathbf{w}$. Show all steps in details specifying which one of the properties $(a), \cdots, (8)$ that you use.

By property (4), there is a vector $-\mathbf{u}$ in V such that $-\mathbf{u} \oplus \mathbf{u} = \mathbf{0}$. Hence

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{u} \oplus \mathbf{w} \Rightarrow -\mathbf{u} \oplus (\mathbf{u} \oplus \mathbf{v}) = -\mathbf{u} \oplus (\mathbf{u} \oplus \mathbf{w}).$$

Using property (2), we then have

$$(-\mathbf{u} \oplus \mathbf{u}) \oplus \mathbf{v} = (-\mathbf{u} \oplus \mathbf{u}) \oplus \mathbf{w}.$$

Using property (4) we have

$$\mathbf{0} \oplus \mathbf{v} = \mathbf{0} \oplus \mathbf{w}$$

and finally, using property (3), we get

$$\mathbf{u} = \mathbf{w}$$

as required.