MATH 3333

Midterm II

October 18, 2007

Name :

I.D. no. :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.

- Show as much work as possible. Answers without explanation will not receive any credit.

- Best of Luck.

i) **(20 Points)** Using the adjoint matrix method, find A^{-1} where

$$A = \begin{pmatrix} 2 & 0 & -1 \\ -3 & 5 & 8 \\ 0 & -4 & -5 \end{pmatrix}$$

$$A_{11} = (-1)^{1+1} \det \begin{bmatrix} 5 & 8 \\ -4 & -5 \end{bmatrix} = 7 \quad A_{12} = (-1)^{1+2} \det \begin{bmatrix} -3 & 8 \\ 0 & -5 \end{bmatrix} = -15$$
$$A_{13} = (-1)^{1+3} \det \begin{bmatrix} -3 & 5 \\ 0 & -4 \end{bmatrix} = 12 \quad A_{21} = (-1)^{2+1} \det \begin{bmatrix} 0 & -1 \\ -4 & -5 \end{bmatrix} = 4$$
$$A_{22} = (-1)^{2+2} \det \begin{bmatrix} 2 & -1 \\ 0 & -5 \end{bmatrix} = -10 \quad A_{23} = (-1)^{2+3} \det \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} = 8$$
$$A_{31} = (-1)^{3+1} \det \begin{bmatrix} 0 & -1 \\ 5 & 8 \end{bmatrix} = 5 \quad A_{32} = (-1)^{3+2} \det \begin{bmatrix} 2 & -1 \\ -3 & 8 \end{bmatrix} = -13$$
$$A_{33} = (-1)^{3+3} \det \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix} = 10$$

Hence we get

$$Adj(A) = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 7 & 4 & 5 \\ -15 & -10 & -13 \\ 12 & 8 & 10 \end{pmatrix}$$

To obtain determinant of A we expand along the first row to get

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 2.$$

Finally, the formula $A^{-1} = \frac{1}{\det(A)} Adj(A)$ gives us

$$A^{-1} = \begin{pmatrix} 7/2 & 2 & 5/2 \\ -15/2 & -5 & -13/2 \\ 6 & 4 & 5 \end{pmatrix}$$

ii) (20 **Points**)

a) Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Determine whether $\mathbf{v} = \begin{pmatrix} 6 \\ -2 \\ 10 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ are in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$.

$a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{v}$ gives us the system of linear equations

$$a + 2b = 6, -b = -2, 3a + 2b = 10 \Rightarrow a = 2, b = -2 \Rightarrow \mathbf{v} \text{ is in } Span\{\mathbf{v}_1, \mathbf{v}_2\}.$$

$a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{w}$ gives us the system of linear equations

$$a + 2b = -2, -b = 3, 3a + 2b = 1$$

The first two equations imply that $a = 4, b = -3$ but these values do not satisfy the third equation. Hence $\mathbf{w}$ does not lie in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$.

b) Let A, B, C be three $n \times n$ matrices such that $AB = AC$. Prove that if $\det(A) \neq 0$ then $B = C$.

$\det(A) \neq 0$ implies that A^{-1} exists. Multiply both sides of $AB = AC$ with A^{-1} to get

$$A^{-1}(AB) = A^{-1}(AC) \Rightarrow B = C$$

as required.

iii) (20 **Points**)

a) Show that the set $S = \{t^2+1, 2t, t+2\}$ spans the vector space P_2 of all polynomials of degree less than or equal to 2.

Let $at^2 + bt + c$ be a vector in P_2. We have to find constants a_1, a_2, a_3 such that

$$a_1(t^2+1) + b_1(2t) + a_3(t+2) = at^2 + bt + c.$$

This gives us the system of linear equations

$$a_1 = a, 2a_2 + a_3 = b, a_1 + 2a_3 = c \Rightarrow a_1 = a, a_2 = \frac{a+2b-c}{4}, a_3 = \frac{c-a}{2}$$

This implies that any vector in P_2 lies in the span of S, hence $Span(S) = P_2$.

b) Let A be a 2×2 matrix such that $A^3 = 3A$. Show that either A is singular or $\det(A) = \pm 3$.

Taking Determinant of both sides of the equation we get

$$\begin{aligned}
&\det(A^3) = \det(3A) \Rightarrow \det(A)^3 = \det(3I_2)\det(A) \Rightarrow \det(A)^3 = 9\det(A) \\
&\Rightarrow \det(A)^3 - 9\det(A) = 0 \Rightarrow \det(A)\Big(\det(A)^2 - 9\Big) = 0 \\
&\Rightarrow \det(A) = 0 \text{ or } \det(A)^2 = 9 \Rightarrow A \text{ is singular or } \det(A) = \pm 3.
\end{aligned}$$

iv) (20 **Points)**

a) Fix a $n \times n$ matrix A. Let W be the subset of the vector space $V = M_{nn}$ consisting of all matrices B that satisfy $AB = BA$. Is W a vector subspace of M_{nn} ? Explain your answer.

Let B_1 and B_2 be two vectors in W. Hence we have $AB_1 = B_1A$ and $AB_2 = B_2A$. We have to check two conditions.

i. **Closure under matrix multiplication :**

$$A(B_1 + B_2) = AB_1 + AB_2 = B_1A + B_2A = (B_1 + B_2)A$$

This implies that $B_1 + B_2$ also lies in W.

ii. **Closure under scalar multiplication :** Let c be a real number. Then

$$A(cB_1) = c(AB_1) = c(B_1A) = (cB_1)A$$

This implies that cB_1 also lies in W.

Hence we can conclude that W is a vector subspace of M_{nn}.

b) Let V be the set of all positive real numbers. Define the operator $\oplus$ by $\mathbf{u} \oplus \mathbf{v} := \mathbf{uv} - 1$ and the operator $\odot$ by $c \odot \mathbf{u} := \mathbf{u}$. Is V a vector space ? Explain your answer.

Consider $\mathbf{u} = 1/2$ and $\mathbf{v} = 1/2$. Both $\mathbf{u}, \mathbf{v}$ lie in V. But $\mathbf{u} \oplus \mathbf{v} = (1/2)(1/2) - 1 = -3/4$. Hence $\mathbf{u} \oplus \mathbf{v}$ does not lie in V. This implies that V is not closed under $\oplus$ and hence V is not a vector space.

v) **(20 Points)** Let

$$A_2 = \begin{bmatrix} x & 1 \\ 1 & x \end{bmatrix}, A_3 = \begin{bmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{bmatrix}, A_4 = \begin{bmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{bmatrix}$$

Show that

$$\det(A_4) = x\det(A_3) - \det(A_2).$$

Expanding $\det(A_4)$ along the first row, we get

$$\begin{aligned} \det(A_4) &= x\det(\begin{bmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{bmatrix}) - 1\det(\begin{bmatrix} 1 & 1 & 0 \\ 0 & x & 1 \\ 0 & 1 & x \end{bmatrix}) \\ &= x\det(A_3) - \Big(1\det(\begin{bmatrix} x & 1 \\ 1 & x \end{bmatrix}) - 1\det(\begin{bmatrix} 0 & 1 \\ 0 & x \end{bmatrix}) + 0\det(\begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix})\Big) \\ &= x\det(A_3) - \det(A_2) \end{aligned}$$

vi) **(Bonus problem : 5 Points)** Let $A_5 = \begin{bmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{bmatrix}$. Show that $\det(A_5) = x\det(A_4) - \det(A_3)$.

You obtain this formula by imitating the calculation we did above - expanding $\det(A_5)$ along the first row.