MATH 3333

Final Exam

December 14, 2007

<u>Name</u> :

<u>I.D. no.</u> :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.

- Show as much work as possible. Answers without explanation will not receive any credit.

- Best of Luck.

i) Consider the function

$$L : R^3 \to R^3 \text{ defined by } L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} 4u_2 \\ 4u_1 + 3u_3 \\ 3u_2 \end{bmatrix}$$

a) (10 points) Show that L is a linear transformation.

b) (10 points) Find a basis for and dimension of the Kernel of L.

c) (10 points) Find a basis for and dimension of the Range of L.

d) (10 points) Let $S = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$ be the standard basis of R^3. Find the matrix representing L with respect to the ordered basis S.

e) (20 points) Let $T = \{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$ be another basis of R^3. Find the matrix representing L with respect to the ordered basis T.

ii) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

a) (15 points) Find the characteristic polynomial and eigenvalues of A.

b) (20 points) Find the eigenspace associated to eigenvalues found in part (a).

c) (15 points) Find a non-singular matrix P which satisfies $P^{-1}AP = D$, where D is a diagonal matrix. Check the identity $P^{-1}AP = D$ by explicit computation.

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iii) (20 points) Use **Cramer's rule** to solve the following system of linear equations

$$2x - y + 4z = 0 \qquad 3y - 3z = 2 \qquad x + y + z = 1$$

iv) (30 points) Determine which of the given subsets forms a basis of R^3. Express the vector $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ as a linear combination of the vectors in each subset that is a basis.

$$(a) \quad \{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}, \qquad\qquad (b) \quad \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} \}$$

v) (20 points) Consider the following right angled triangle with sides a, b, c.

Consider the matrix $A = \begin{bmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & 0 \end{bmatrix}$. Show that the eigenvalues of A are $0, c, -c$.

vi) (20 points) Let $A = \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{bmatrix}$. Show that

$$\det(A) = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d).$$

Hint : It is easier if you simplify the matrix before you start taking determinants. Also, you might find the following factorization formulas useful : $x^2 - y^2 = (x-y)(x+y)$ and $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.