Math 3113, Quiz IV, Solution

1. Consider the initial value problem

$$tx''' + 2x'' + (t-1)x' = 0 \qquad x(0) = x'(0) = x''(0) = 0.$$

   If $\mathcal{L}\{x(t)\} = X(s)$ then obtain the first order differential equation satisfied by $X(s)$. Write it in the separable form but **DO NOT** solve it.

   We have $\mathcal{L}\{tx'''\} = -(s^3X(s))' = -(3s^2X(s)+s^3X'(s)), \mathcal{L}\{2x''\} = 2s^2X(s)$ and $\mathcal{L}\{(t-1)x'\} = \mathcal{L}\{tx'\} - \mathcal{L}\{x'\} = -(sX(s))' - sX(s) = -(X(s) + sX'(s)) - sX(s)$. Hence applying Laplace Transform to the differential equation and collecting terms involving $X(s)$ and $X'(s)$ we get

$$X'(s)(-s^3 - s) + X(s)(-s^2 - s - 1) = 0 \Rightarrow \frac{X'(s)}{X(s)} = -\frac{s^2+s+1}{s^3+s}$$

   (Note that if you solve the above separable DE you get that $X(s)$ equals the expression given in problem 2 below. Hence the 2 problems are parts of just one problem.)

2. Find the Laplace inverse of

$$X(s) = \frac{1}{s(s^2+1)}$$

   There are several ways to solve this :

   (a) Using Convolution product Theorem :

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s^2+1}\} * \mathcal{L}^{-1}\{\frac{1}{s}\} = \sin t * 1 = \int_0^t \sin\tau d\tau = 1 - \cos t.$$

   (b) Using Partial Fractions :

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1} \Rightarrow \mathcal{L}^{-1}\{X(s)\} = 1 - \cos t.$$

   (c) Using Transform formula for integrals :

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} = \mathcal{L}^{-1}\{\frac{\frac{1}{s^2+1}}{s}\} = \int_0^t \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}d\tau = \int_0^t \sin\tau d\tau = 1 - \cos t.$$

   Hence, the solution is

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 1 - \cos t.$$