MATH 3113

Midterm I, Form B, SOLUTIONS

March 2, 2007

Name :

I.D. no. :

- Calculators are not allowed. The problems are set so that you should not need calculators at all.

- Show as much work as possible. Answers without explanation will not receive any credit.

- Best of Luck.

i) (15 Points) Find the general solution of the differential equation

$$x^2 \frac{dy}{dx} = 5x^5y^2 + 4y^2x^4$$

Divide both sides of the equation with x^2y^2 to get the separable differential equation

$$\begin{aligned}
\frac{1}{y^2}\frac{dy}{dx} &= 5x^3 + 4x^2 \\
\int \frac{1}{y^2} dy &= \int (5x^3 + 4x^2)dx + C \\
-\frac{1}{y} &= \frac{5}{4}x^4 + \frac{4}{3}x^3 + C \\
y(x) &= \frac{-1}{\frac{5}{4}x^4 + \frac{4}{3}x^3 + C}
\end{aligned}$$

ii) (15 Points) Show that the following differential equation is exact and then solve it

$$(5x^4y^5 + 2y^2)dx + (5x^5y^4 + 4xy + 3y)dy = 0.$$

We have $M = 5x^4y^5 + 2y^2$ and $N = 5x^5y^4 + 4xy + 3y$. To show exactness we have to show that $M_y = N_x$.

$$M_y = 25x^4y^4 + 4y, \qquad N_x = 25x^4y^4 + 4y.$$

Now, we first use $F_x = M$ and integrate with respect to x treating y to be fixed constant and get

$$F(x, y) = x^5y^5 + 2y^2x + g(y)$$

Using $F_y = N$ we get $g'(y) = 3y$ which implies $g(y) = 3y^2/2$. Hence the implicit solution of the differential equation is

$$x^5y^5 + 2y^2x + 3y^2/2 = C.$$

iii) (15 Points) A tank contains 3000 liters of a solution consisting of 200 Kg. of salt dissolved in water. Pure water is pumped into the tank at the rate of 30 liters per second, and the mixture is pumped out at the same rate. Find a formula for the amount of salt in the tank at time t.

We have $r_i = 30 = r_o, c_i = 0$, since the water pumped in is pure water, $V_0 = 3000$ and $x(0) = 200$. Since $V(t) = V_0 + (r_i - r_o)t$, we get $V(t) = 3000$. The differential equation satisfied by $x(t)$ is

$$x' = r_i c_i - r_o c_o(t) = r_i c_i - r_o x(t)/V(t)$$

which gives us

$$\begin{aligned} x' &= -30\frac{x}{3000} \implies x' = -\frac{x}{100} \\ \int \frac{1}{x}dx &= \int -\frac{1}{100}dt + C \implies \ln(x) = \frac{t}{100} + C \\ x(t) &= Ae^{-t/100} \text{ where } A = e^C \end{aligned}$$

We use the initial condition $x(0) = 200$ to get $A = 200$. Hence the final answer is

$$x(t) = 200e^{-t/100}$$

iv) (15 Points) Find the general solution of the system of differential equations

$$\begin{aligned} x' &= 2y \\ y' &= 3y - \frac{5}{2}x \end{aligned}$$

Differentiate both sides of the first equation with respect to t and use the second equation to get $x'' = y' = 6y - 5x$. Again using the first equation we get

$$x'' - 3x' + 5x = 0$$

The characteristic equation is $r^2 - 3r + 5 = 0$ whose roots are $r = \frac{3}{2} + i\frac{\sqrt{11}}{2}, r = \frac{3}{2} - i\frac{\sqrt{11}}{2}$. The corresponding general solution is

$$x(t) = e^{\frac{3}{2}t}(C_1 \cos(\frac{\sqrt{11}}{2}t) + C_2 \sin(\frac{\sqrt{11}}{2}t))$$

Again using the first equation $y = x'/2$ we obtain using product rule

$$\begin{aligned} y(t) = \frac{1}{2}(\frac{3}{2}e^{\frac{3}{2}t}(C_1 \cos(\frac{\sqrt{11}}{2}t) + C_2 \sin(\frac{\sqrt{11}}{2}t)) + \\ e^{\frac{3}{2}t}(-\frac{\sqrt{11}}{2}C_1 \sin(\frac{\sqrt{11}}{2}t) + \frac{\sqrt{11}}{2}C_2 \cos(\frac{\sqrt{11}}{2}t))) \end{aligned}$$

v) (20 Points) Solve the initial value problem

$$y'' - 5y' = 10, \qquad y(0) = 2, \;\; y'(0) = 3.$$

This is a non-homogeneous differential equation with constant coefficients and hence the general solution is of the form

$$y(x) = y_c(x) + y_p(x)$$

Computation of y_c : The associated homogeneous D.E. is $y'' - 5y' = 0$ whose characteristic equation is $r^2 - 5r = 0$. The roots are $r = 0, r = 5$ and hence we get

$$y_c(x) = C_1 + C_2 e^{5x}$$

Computation of y_p : Here $f(x) = 10$ and hence the first guess is $y_p = A$. But this lead to duplication and hence we have to make the second guess by multiplying with $x : y_p(x) = Ax$. We substitute this in the original differential equation to find the value of A. Since $y_p' = A$ and $y_p'' = 0$ we get $0 - 5(A) = 10 \implies A = -2$ and hence

$$y_p(x) = -2x$$

Combining these two we get the general solution

$$y(x) = C_1 + C_2 e^{5x} - 2x$$

To find the values of C_1, C_2 we have to use the initial value conditions. This gives us

$$2 = y(0) = C_1 + C_2 \text{ and } 3 = y'(0) = 5C_1 - 2$$

and hence $C_1 = 1, C_2 = 1$. The final answer is

$$y(x) - 1 + e^{5x} - 2x.$$

vi) (20 Points) Use variation of parameters to find the particular solution y_p of the differential equation

$$y'' + 16y = 2\sec(4x)$$

<u>Computation of y_c</u> : The associated homogeneous D.E. is $y'' + 16y' = 0$ whose characteristic equation is $r^2 + 16 = 0$. The roots are $r = 4i, r = -4i$ and hence we get

$$y_c(x) = C_1 \cos(4x) + C_2 \sin(4x)$$

The method of variation of parameters tells us that the particular solution is of the form

$$y_p(x) = u_1(x)\cos(4x) + u_2(x)\sin(4x)$$

where the functions u_1 and u_2 satisfy the conditions

$$\begin{aligned} u_1' \cos(4x) + u_2' \sin(4x) &= 0 \\ -u_1' 4\sin(4x) + u_2' 4\cos(4x) &= 2\sec(4x) \end{aligned}$$

Solving for u_1' and u_2' we get

$$u_1'(x) = -\frac{1}{2}\tan(4x) \text{ and } u_2'(x) = \frac{1}{2}$$

Integrating these we get

$$u_1(x) = -\frac{1}{8}\ln|\sec(4x)| \text{ and } u_2(x) = \frac{x}{2}$$

Substituting these we get the final answer

$$y_p(x) = -\frac{1}{8}\ln|\sec(4x)|\cos(4x) + \frac{x}{2}\sin(4x).$$