MULTIPLICITY ONE FOR L-FUNCTIONS AND APPLICATIONS

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Abstract. We give conditions for when two Euler products are the same given that they satisfy a functional equation and their coefficients satisfy a partial Ramanujan bound and do not differ by too much. Additionally, we prove a number of multiplicity one type results for the number-theoretic objects attached to L-functions. These results follow from our main result about L-functions.

1. Introduction

An L-function is a Dirichlet series that converges absolutely in some right half plane, has a meromorphic continuation to a function of order 1, with finitely many poles, satisfies a functional equation, and admits an Euler product. For example, the (incomplete) L-functions attached to tempered, cuspidal automorphic representations, or the Hasse-Weil L-functions attached to non-singular, projective, algebraic varieties defined over a number field, conjecturally satisfy these conditions.

In this paper, using standard techniques from analytic number theory, we prove a strong multiplicity one result for such L-functions (without reference to any underlying automorphic or geometric object). We closely follow the work [10] and we redo their arguments for two reasons. First, our results are more general in that they have slightly weaker hypotheses. Second, we think that the techniques should be better known, especially to those who study L-functions automorphically.

One of the defining axioms for the class of L-functions we consider is the existence of an Euler product. There exists a number d, called the degree of the L-function, such that the local Euler factor is of the form $Q_p(p^{-s})^{-1}$, where $Q_p(X)$ is a polynomial satisfying $Q_p(0) = 1$, and $Q_p(X)$ has degree d for almost all primes. We say that a given L-function *satisfies the Ramanujan conjecture*, if the roots of Q_p are of absolute value at least 1, for all p.

The multiplicity one results we discuss in this paper are statements which assert that if two L-functions are sufficiently close, then they must be equal. A model example is:

Theorem 1.1. *Suppose $L_1(s) = \sum a_1(n)n^{-s}$ and $L_2(s) = \sum a_2(n)n^{-s}$ are Dirichlet series which continue to meromorphic functions of order 1 satisfying appropriate functional equations and having appropriate Euler products. Assume that $L_1(s)$ and $L_2(s)$ satisfy the Ramanujan conjecture. Assume also that $a_1(p) = a_2(p)$ for almost all p. Then $L_1(s) = L_2(s)$.*

The precise conditions on the functional equation and Euler product are described in Section 2.1. A weaker version of Theorem 1.1, requiring equality of the

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local Euler factors instead of the pth Dirichlet coefficients, is given in [20]. Theorem 1.1 is also a consequence of the main result in [10]. The result we will actually prove, Theorem 2.2, is stronger. First, instead of requiring equality of the pth Dirichlet series coefficients, we only require that they are close on average. Second, the Ramanujan hypothesis can be slightly relaxed.

We will present three applications of strong multiplicity one for L-functions. The first application is to cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, where $\mathbb{A}_{\mathbb{Q}}$ denotes the ring of adeles of the number field $\mathbb{Q}$. Any such representation π factors as $\pi = \otimes \pi_p$, where π_p is an irreducible admissible representation of $\mathrm{GL}(n, \mathbb{Q}_p)$ (we mean $\mathbb{Q}_p = \mathbb{R}$ for $p = \infty$). Attached to π is an automorphic L-function $L(s, \pi)$, whose finite part is $L_{\mathrm{fin}}(s, \pi) = \prod_{p<\infty} L(s, \pi_p)$. The completion of $L_{\mathrm{fin}}(s, \pi)$ is known to be "nice", and hence $L_{\mathrm{fin}}(s, \pi)$ is the kind of function to which Theorem 1.1 applies. At almost all primes p we have $L(s, \pi_p) = \det(1 - A(\pi_p)p^{-s})^{-1}$, where $A(\pi_p) = \mathrm{diag}(\alpha_{1,p}, \ldots, \alpha_{n,p})$ is a diagonal matrix whose entries are the Satake parameters at p. The Ramanujan conjecture is the assertion that each π_p is tempered, which in this context implies that $|\alpha_{j,p}| = 1$. In particular note that $L(s, \pi_p)$ is a polynomial in p^{-s} and this polynomial has all its roots on the unit circle.

An easy consequence of Theorem 1.1 is the following.

Theorem 1.2. *Suppose that π and π' are (unitary) cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ satisfying $\mathrm{tr}(A(\pi_p)) = \mathrm{tr}(A(\pi'_p))$ for almost all p. Assume that both $L_{\mathrm{fin}}(s, \pi)$ and $L_{\mathrm{fin}}(s, \pi')$ satisfy the Ramanujan conjecture. Then $\pi = \pi'$.*

Most statements of strong multiplicity one in the literature are phrased in terms of $A(\pi_p)$ and $A(\pi'_p)$ being conjugate, instead of the much weaker condition of the equality of their traces. Using the stronger version of Theorem 1.1, we will in fact prove a stronger result which only requires that the traces are close enough on average; see Theorem 3.1 for the precise statement.

Our second application is to Siegel modular forms of degree 2. For such modular forms Weissauer [27] has proved the Ramanujan conjecture. The Dirichlet coefficients $a_i(p)$ appearing in Theorem 1.1 are essentially the Hecke eigenvalues for the Hecke operator $T(p)$. We therefore have the following:

Theorem 1.3. *Suppose F_j, for $j = 1, 2$, are Siegel Hecke eigenforms of weight k_j for $\mathrm{Sp}(4, \mathbb{Z})$, with Hecke eigenvalues $\mu_j(n)$. If $p^{3/2-k_1}\mu_1(p) = p^{3/2-k_2}\mu_2(p)$ for all but finitely many p, then $k_1 = k_2$ and F_1, F_2 have the same eigenvalues for the Hecke operator $T(n)$ for all n.*

The remarkable fact here is that the Hecke operator $T(p)$ alone does not generate the local Hecke algebra at p. This Hecke algebra is generated by $T(p)$ and $T(p^2)$. The fact that the coincidence of the eigenvalues for $T(p)$ is enough is of course a global phenomenon. Using the result of [21], we see that if Böcherer's conjecture is true then not only are the Hecke eigenvalues of F_1, F_2 in Theorem 1.3 equal but we get $F_1 = F_2$. Again, using the averaged version of Theorem 2.2, we can prove a stronger result; see Theorem 3.2.

Our third application concerns the Hasse-Weil zeta function of hyperelliptic curves; see Proposition 3.3. This Proposition is in a similar spirit to those mentioned above. Assuming the L-functions satisfy a functional equation of a form they are expected to satisfy, we can apply our analytic theorems to prove a result about the underlying (in this case) geometric object.

Notation. We review some notation from analytic number theory for completeness. Given two functions $f(x)$, $g(x)$,

- we write $f(x) \sim g(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/g(x) = 1$;
- we write $f(x) \ll g(x)$ as $x \to \infty$ if there exists $C > 0$ and $x_0 > 0$ so that if $x > x_0$ then $|f(x)| \le C|g(x)|$; this is also written as $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$.

In this paper we drop the phrase "as $x \to \infty$" when using the above notation.

2. Multiplicity one for L-functions

In this section we describe the L-functions for which we will prove a multiplicity one result. As in other approaches to L-functions viewed from a classical perspective, such as that initiated by Selberg [23], we consider Dirichlet series with a functional equation and an Euler product. However, in contrast to Selberg, we strive to make all our axioms as specific as possible. Presumably (as conjectured by Selberg) these different axiomatic approaches all describe the same objects: $L(s,\pi)$ where π is a cuspidal automorphic representation of $\mathrm{GL}(n)$.

2.1. L-function background.

Before getting to L-functions, we recall two bits of terminology that will be used in the following discussion. An entire function $f : \mathbb{C} \to \mathbb{C}$ is said to have order at most α if for all $\epsilon > 0$:

$$f(s) = \mathcal{O}(\exp(|s|^{\alpha+\epsilon})).$$

Moreover, we say f has order equal to α if f has order at most α, and f does not have order at most γ for any $\gamma < \alpha$. The notion of order is relevant because functions of finite order admit a factorization as described by the Hadamard Factorization Theorem and the Γ-function and L-functions are all of order 1.

In order to ease notation, we use the normalized Γ-functions defined by:

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\,\Gamma(s/2) \qquad \text{and} \qquad \Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\,\Gamma(s).$$

An L-function is a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \tag{2.1}$$

where $s = \sigma + it$ is a complex variable. We assume that $L(s)$ converges absolutely in the half-plane $\sigma > 1$ and has a meromorphic continuation to all of $\mathbb{C}$. The resulting function is of order 1, admitting at most finitely many poles, all of which are located on the line $\sigma = 1$. Finally, $L(s)$ must have an Euler product and satisfy a functional equation as described below.

The functional equation involves the following parameters: a positive integer N, complex numbers $\mu_1, \ldots, \mu_J$ and $\nu_1, \ldots, \nu_K$, and a complex number ε. The completed L-function

$$\Lambda(s) := N^{s/2} \prod_{j=1}^{J} \Gamma_{\mathbb{R}}(s+\mu_j) \prod_{k=1}^{K} \Gamma_{\mathbb{C}}(s+\nu_k) \cdot L(s) \tag{2.2}$$

is a meromorphic function of finite order, having the same poles as $L(s)$ in $\sigma > 0$, and satisfying the functional equation

$$\Lambda(s) = \varepsilon \overline{\Lambda}(1-s). \tag{2.3}$$

The number $d = J + 2K$ is called the *degree* of the L-function.

We require some conditions on the parameters μ_j and ν_j. The *temperedness condition* is the assertion that $\Re(\mu_j) \in \{0,1\}$ and $\Re(\nu_j)$ a positive integer or half-integer. With those restrictions, there is only one way to write the parameters in the functional equation, as proved in Proposition 2.1. This restriction is not known to be a theorem for most automorphic L-functions. In order to state theorems which apply in those cases, we will make use of a "partial Selberg bound," which is the assertion that $\Re(\mu_j),\ \Re(\nu_j) > -\frac{1}{2}$.

The Euler product is a factorization of the L-function into a product over the primes:

$$L(s) = \prod_p F_p(p^{-s})^{-1}, \tag{2.4}$$

where F_p is a polynomial of degree at most d:

$$F_p(z) = (1 - \alpha_{1,p} z) \cdots (1 - \alpha_{d,p} z). \tag{2.5}$$

If $p|N$ then p is a *bad* prime and the degree of F_p is strictly less than d, in other words, $\alpha_{j,p} = 0$ for at least one j. Otherwise, p is a *good* prime, in which case the $\alpha_{j,p}$ are called the *Satake parameters* at p. The *Ramanujan bound* is the assertion that at a good prime $|\alpha_{j,p}| = 1$, and at a bad prime $|\alpha_{j,p}| \le 1$.

The Ramanujan bound has been proven in very few cases, the most prominent of which are holomorphic forms on GL(2) and GSp(4). See [22] for a survey of what progress is known towards proving the Ramanujan bound. Also see [6].

We write $|\alpha_{j,p}| \le p^\theta$, for some $\theta < \frac{1}{2}$, to indicate progress toward the Ramanujan bound, referring to this as a "partial Ramanujan bound."

We will need to use symmetric and exterior power L-functions associated to a L-function $L(s)$. Let S be the finite set of *bad* primes p of $L(s)$. The partial symmetric and exterior square L-functions are defined as follows.

$$L^S(s, \mathrm{sym}^n) = \prod_{p \notin S} \prod_{i_1 + \ldots + i_d = n} (1 - \alpha_{1,p}^{i_1} \ldots \alpha_{d,p}^{i_d} p^{-s})^{-1} \tag{2.6}$$

$$L^S(s, \mathrm{ext}^n) = \prod_{p \notin S} \prod_{1 \le i_1 < \ldots < i_n \le d} (1 - \alpha_{i_1,p} \ldots \alpha_{i_n,p} p^{-s})^{-1}. \tag{2.7}$$

We do not define the local Euler factors at the bad primes since there is no universal recipe for these. It is conjectured that the symmetric and exterior power L-functions are in fact L-functions in the sense described above. In that case, Proposition 2.1 tells us that the bad Euler factors are uniquely determined. For applications that we present in this paper, the partial L-functions suffice.

In most cases it is not necessary to specify the local factors at the bad primes because, by almost any version of the strong multiplicity one theorem, an L-function is determined by its Euler factors at the good primes. For completeness we state a simple version of the result.

In the following proposition we use the term "L-function" in a precise sense, referring to a Dirichlet series which satisfies a functional equation of the form (2.2)-(2.3) with the restrictions $\Re(\mu_j) \in \{0,1\}$ and $\Re(\nu_j)$ a positive integer or half-integer, and having an Euler product satisfying (2.4)-(2.5). We refer to the quadruple $(d, N, (\mu_1, \ldots, \mu_J : \nu_1, \ldots, \nu_K), \varepsilon)$ as the *functional equation data* of the L-function.

Proposition 2.1. *Suppose that* $L_j(s) = \prod_p F_{p,j}(p^{-s})^{-1}$, *for* $j = 1, 2$, *are* L-*functions which satisfy a partial Ramanujan bound for some* $\theta < \frac{1}{2}$. *If* $F_{p,1} = F_{p,2}$ *for all but finitely many* p, *then* $F_{p,1} = F_{p,2}$ *for all* p, *and* L_1 *and* L_2 *have the same functional equation data.*

In particular, the proposition shows that the functional equation data of an L-function is well defined. There are no ambiguities arising, say, from the duplication formula of the Γ-function. Also, we remark that the partial Ramanujan bound is essential. One can easily construct counterexamples to the above proposition using Saito-Kurokawa lifts, which do not satisfy the partial Ramanujan bound.

Proof. Let $\Lambda_j(s)$ be the completed L-function of $L_j(s)$ and consider

$$\begin{aligned}\lambda(s) &= \frac{\Lambda_1(s)}{\Lambda_2(s)}\\ &= \left(\frac{N_1}{N_2}\right)^{s/2} \frac{\prod_j \Gamma_\mathbb{R}(s+\mu_{j,1}) \prod_k \Gamma_\mathbb{C}(s+\nu_{k,1})}{\prod_j \Gamma_\mathbb{R}(s+\mu_{j,2}) \prod_k \Gamma_\mathbb{C}(s+\nu_{k,2})} \prod_p \frac{F_{p,1}(p^{-s})^{-1}}{F_{p,2}(p^{-s})^{-1}}.\end{aligned} \tag{2.8}$$

By the assumption on $F_{p,j}$, the product over p is really a finite product. Thus, (2.8) is a valid expression for $\lambda(s)$ for all s.

By the partial Ramanujan bound and the assumptions on μ_j and ν_j, we see that $\lambda(s)$ has no zeros or poles in the half-plane $\Re(s) > \theta$. But by the functional equations for L_1 and L_2 we have $\lambda(s) = (\varepsilon_1/\varepsilon_2)\overline{\lambda}(1-s)$. Thus, $\lambda(s)$ also has no zeros or poles in the half-plane $\Re(s) < 1-\theta$. Since $\theta < \frac{1}{2}$, we conclude that $\lambda(s)$ has no zeros or poles in the entire complex plane.

If the product over p in (2.8) were not empty, then the fact that $\{\log(p)\}$ is linearly independent over the rationals implies that $\lambda(s)$ has infinitely many zeros or poles on some vertical line. Thus, $F_{p,1} = F_{p,2}$ for all p.

The Γ-factors must also cancel identically, because the right-most pole of $\Gamma_\mathbb{R}(s+\mu)$ is at $-\mu$, and the right-most pole of $\Gamma_\mathbb{C}(s+\nu)$ is at $-\nu$. This leaves possible remaining factors of the form $\Gamma_\mathbb{C}(s+1)/\Gamma_\mathbb{R}(s+1)$, but that also has poles because the $\Gamma_\mathbb{R}$ factor cancels the first pole of the $\Gamma_\mathbb{C}$ factor, but not the second pole. Note that the restriction $\Re(\mu) \in \{0, 1\}$ is a critical ingredient in this argument.

This leaves the possibility that $\lambda(s) = (N_1/N_2)^{s/2}$, but such a function cannot satisfy the functional equation $\lambda(s) = (\varepsilon_1/\varepsilon_2)\overline{\lambda}(1-s)$ unless $N_1 = N_2$ and $\varepsilon_1 = \varepsilon_2$. □

2.2. **The strong multiplicity one theorem for L-functions.** In this section we state a version of strong multiplicity one for L-functions which is stronger than Proposition 2.1 because it only requires the Dirichlet coefficients $a(p)$ and $a(p^2)$ to be reasonably close. This is a significantly weaker condition than equality of the local factor.

Although the main ideas behind the proof appear in Kaczorowski-Perelli [10] and Soundararajan [26], we give a slightly stronger version which assumes a partial Ramanujan bound $\theta < \frac{1}{6}$, plus an additional condition, instead of the full Ramanujan conjecture. We provide a self-contained account because we also wish to bring awareness of these techniques to people with a more representation-theoretic approach to L-functions.

Theorem 2.2. *Suppose $L_1(s)$, $L_2(s)$ are Dirichlet series with Dirichlet coefficients $a_1(n)$, $a_2(n)$, respectively, which continue to meromorphic functions of order 1 satisfying functional equations of the form* (2.2)-(2.3) *with a partial Selberg bound $\Re(\mu_j)$, $\Re(\nu_j) > -\frac{1}{2}$ for both functions, and having Euler products satisfying* (2.4)-(2.5)*. Assume a partial Ramanujan bound for some $\theta < \frac{1}{6}$ holds for both functions, and that the Dirichlet coefficients at the primes are close to each other in the sense that*

$$\sum_{p \le X} p \log(p)|a_1(p) - a_2(p)|^2 \ll X. \tag{2.9}$$

We have $L_1(s) = L_2(s)$ if either of the following two conditions are satisfied

1) $\displaystyle\sum_{p \le X} |a_1(p^2) - a_2(p^2)|^2 \log p \ll X.$

2) For each of $L_1(s)$ and $L_2(s)$, separately, any one of the following holds:
  - *a) The Ramanujan bound $\theta = 0$.*
  - *b) The partial symmetric square (2.6) of the function has a meromorphic continuation past the $\sigma = 1$ line, and only finitely many zeros or poles in $\sigma \ge 1$.*
  - *c) The partial exterior square (2.7) of the function has a meromorphic continuation past the $\sigma = 1$ line, and only finitely many zeros or poles in $\sigma \ge 1$.*

Note that condition (2.9) is satisfied if $|a_1(p) - a_2(p)| \ll 1/\sqrt{p}$, in particular, if $a_1(p) = a_2(p)$ for all but finitely many p, or more generally if $a_1(p) = a_2(p)$ for all but a sufficiently thin set of primes. In particular, $a_1(p)$ and $a_2(p)$ can differ at infinitely many primes. Also, by the prime number theorem [2, Theorem 4.4] in the form

$$\sum_{p < X} \log(p) \sim X, \tag{2.10}$$

condition 2a) for both L-functions implies condition 1).

The condition $\theta < \frac{1}{6}$ arises from the p^{-3s} terms in the proof of Lemma 2.4. Those terms do not seem to give rise to a naturally occuring L-function at $3s$, so it may be difficult to replace the $\theta < \frac{1}{6}$ condition by a statement about the average of certain Dirichlet coefficients.

2.3. **Some technical lemmas.** In this section we provide the lemmas required for the proof of Theorem 2.2. There are two types of lemmas we require. The first deals with manipulating Euler products and establishing zero-free half-planes via the convergence of those products. The second deals with possible zeros at the edge of the half-plane of convergence.

2.3.1. *Coefficients of related L-functions.* If $L(s) = \sum a(n)n^{-s}$ then for $\rho = \mathrm{sym}^n$ or ext^n we write

$$L(s, \rho) = \sum_j a(j, \rho)\, j^{-s}. \tag{2.11}$$

Lemma 2.3. *If p is a good prime then*
- $a(p, \mathrm{sym}^n) = a(p^n)$,
- $a(p, \mathrm{ext}^2) = a(p)^2 - a(p^2)$,
- $a(p, \mathrm{ext}^3) = a(p^3) + a(p)^3 - 2a(p)a(p^2)$, *and*
- $a(p^2, \mathrm{sym}^2) = a(p^4) - a(p)a(p^3) + a(p^2)^2$.

Proof. Let p be a good prime. Expanding the Euler factor $L_p(s)$ for $L(s)$ we have

$$L_p(s) = \prod_{j=1}^{d} \frac{1}{(1-\alpha_{j,p}\,p^{-s})} = \prod_{j=1}^{d}\sum_{\ell=0}^{\infty} \alpha_{j,p}^{\ell}\,p^{-\ell s} = \sum_{\ell=0}^{\infty} p^{-\ell s} \sum_{n_1+\cdots+n_d=\ell} \alpha_{1,p}^{n_1}\cdots\alpha_{d,p}^{n_d}\,, \tag{2.12}$$

where the n_j are restricted to non-negative integers. Expanding the Euler factor for $L^S(s,\mathrm{sym}^n)$ we have

$$\begin{aligned} L_p(s,\mathrm{sym}^n) &= \prod_{i_1+\cdots+i_d=n} (1-\alpha_{1,p}^{i_1}\cdots\alpha_{d,p}^{i_d}p^{-s})^{-1} \\ &= \prod_{i_1+\cdots+i_d=n}\sum_{\ell=0}^{\infty}\left(\alpha_{1,p}^{i_1}\cdots\alpha_{d,p}^{i_d}\right)^{\ell} p^{-\ell s}. \end{aligned} \tag{2.13}$$

The coefficient of p^{-s} in (2.13) is

$$\sum_{i_1+\cdots+i_d=n} \alpha_{1,p}^{i_1}\cdots\alpha_{d,p}^{i_d}, \tag{2.14}$$

which equals the coefficient of p^{-ns} in (2.12).

The other identities in the lemma just involve expanding the definitions and checking particular coefficients. □

2.3.2. *Manipulating L-functions.* The next lemma tells us that if there are zeros in the critical strip for $\sigma \geq \frac{1}{2}$, these zeros come from Euler factors involving the coefficients $a(p)$ or $a(p^2)$ of the Dirichlet series or the Euler factors of the symmetric or exterior square L-functions.

Lemma 2.4. *Suppose*

$$\begin{aligned} L(s) &= \sum_n a_n n^{-s} \\ &= \prod_{p \text{ bad}}\prod_{j=1}^{d_p}(1-\alpha_{j,p}p^{-s})^{-1}\prod_{p \text{ good}}\prod_{j=1}^{d}(1-\alpha_{j,p}p^{-s})^{-1}, \end{aligned} \tag{2.15}$$

where $|\alpha_{j,p}| \leq p^{\theta}$ for some $\theta \in \mathbb{R}$. Then for $\sigma > 1+\theta$,

$$\begin{aligned} L(s) &= \prod_p (1+a(p)p^{-s}) \cdot \prod_p (1+a(p^2)p^{-2s}) \cdot h_0(s), \\ &= \prod_p (1+a(p)p^{-s}) \cdot L^S(2s,\mathrm{sym}^2)\cdot h_1(s), \\ &= \prod_p (1+a(p)p^{-s}+a(p)^2p^{-2s}) \cdot L^S(2s,\mathrm{ext}^2)^{-1}\cdot h_2(s), \end{aligned} \tag{2.16}$$

where $h_j(s)$ is regular and nonvanishing for $\sigma > \frac{1}{3}+\theta$.

Proof. We can write the Euler product in the form

$$L(s) = \prod_p \sum_{j=0}^{\infty} a(p^j)p^{-js}, \tag{2.17}$$

where

$$a(p^j) \ll_j p^{j\theta} \ll p^{j(\theta+\varepsilon)}, \tag{2.18}$$

with the implied constant depending only on ε. We manipulate the Euler product, introducing coefficients A_j, and B_j where $A_j(p), B_j(p) \ll_j p^{j\theta} \ll p^{j(\theta+\varepsilon)}$. We have

$$\begin{aligned}
L(s) &= \prod_p \sum_{j=0}^{\infty} a(p^j)p^{-js} \\
&= \prod_p \Big(1 + a(p)p^{-s} + A_2(p)p^{-2s} + \sum_{j=3}^{\infty} A_j(p)p^{-js}\Big) \\
&\qquad \times \Big(1 + (a(p^2) - A_2(p))p^{-2s} + \sum_{j=2}^{\infty} B_{2j}(p)p^{-2js}\Big) \\
&\qquad \times \big(1 + O(p^{3\theta}p^{-3s})\big) \qquad (2.19) \\
&= F_1(s)F_2(s)F_3(s), \qquad (2.20)
\end{aligned}$$

say. Note that by the assumptions on A_j and B_j we have

$$\sum_{j=3}^{\infty} A_j(p)p^{-js} = O(p^{3\theta}p^{-3s}) \qquad \text{and} \qquad \sum_{j=2}^{\infty} B_{2j}(p)p^{-2js} = O(p^{4\theta}p^{-4s}). \qquad (2.21)$$

Combining this with (2.18) justifies (2.19).

For the first assertion, set $A_j(p) = 0$ and $B_j(p) = 0$ and note that $F_1(s)$ converges absolutely for $\sigma > 1 + \theta$ and $F_2(s)$ converges absolutely for $\sigma > \frac{1}{2} + \theta$, and $F_3(s)$ converges absolutely for $\sigma > \frac{1}{3} + \theta$.

For the second assertion, set $A_j(p) = 0$. For good primes p, choose $B_j(p)$ so that $F_2(s) = L^S(2s, \mathrm{sym}^2)$. For bad primes p, choose $B_j(p) = 0$. Note that (by the construction of the symmetric square) this choice of B_j satisfies the required bounds. The finitely many factors at the bad primes together with $F_3(s)$ give $h_1(s)$.

For the third assertion, the only modification is to set $A_2(p) = a(p)^2$, $A_j(p) = 0$ for $j \geq 3$, and use the second identity in Lemma 2.3 and appropriate choices for $B_j(p)$ so that $F_2(s) = L^S(2s, \mathrm{ext}^2)^{-1}$. □

2.3.3. *Zeros at the edge of the half-plane of convergence.* The absolute convergence of an Euler product in a half-plane $\sigma > \sigma_0$ implies that the function has no zeros or poles in that region. If the Euler product has a meromorphic continuation to a larger region, it could possibly have zeros or poles on the σ_0-line. The lemma in this section, which is standard and basically follows the proof of Lemma 1 of [10], says that if the Dirichlet coefficients $a(p)$ are small on average then there are finitely many zeros or poles on the σ_0-line. Our modification is that we only require the L-function to satisfy a partial Ramanujan bound.

Note that the lemma is stated with $\sigma_0 = 1$ as the boundary of convergence. Applying the lemma in contexts with a different line of convergence, as in the proof of Theorem 2.2, just involves a simple change of variables $s \to s + A$.

Lemma 2.5. *Let*

$$L(s) = \prod_p \sum_{j=0}^{\infty} a(p^j)p^{-js} \qquad (2.22)$$

and suppose there exists $M_1, M_2 \geq 0$ *and* $\theta < \frac{2}{3}$ *so that* $|a(p^j)| \ll p^{\theta j}$ *and*

$$\sum_{p \leq X} |a(p)|^2 \log p \leq M_1^2 X + o(X), \tag{2.23}$$

$$\sum_{p \leq X} p^{-2}|a(p^2)|^2 \log p \leq M_2^2 X + o(X). \tag{2.24}$$

Then $L(s)$ *is a nonvanishing analytic function in the half-plane* $\sigma > 1$*. Furthermore, if* $L(s)$ *has a meromorphic continuation to a neighborhood of* $\sigma \geq 1$*, then* $L(s)$ *has at most* $(M_1 + 2M_2)^2$ *zeros or poles on the* $\sigma = 1$ *line.*

Note that, by the prime number theorem (2.10), the condition (2.23) on $a(p)$ is satisfied if $|a(p)| \leq M_1$. Also, if $\theta < \frac{1}{2}$ then condition (2.24) on $a(p^2)$ holds with $M_2 = 1$.

The proof of Lemma 2.5 is in Section 2.5.

2.4. **Proof of Theorem 2.2.** Now we have the ingredients to prove Theorem 2.2. The proof begins the same as that of Proposition 2.1, by considering the ratio of completed L-funtions:

$$\lambda(s) := \frac{\Lambda_1(s)}{\Lambda_2(s)}, \tag{2.25}$$

which is a meromorphic function of order 1 and satisfies the functional equation $\lambda(s) = \varepsilon\overline{\lambda}(1-s)$, where $\varepsilon = \varepsilon_1/\varepsilon_2$.

Lemma 2.6. $\lambda(s)$ *has only finitely many zeros or poles in the half-plane* $\sigma \geq \frac{1}{2}$*.*

Assuming the lemma, we complete the proof of Theorem 2.2 as follows. By the functional equation, $\lambda(s)$ has only finitely many zeros or poles, so by the Hadamard factorization theorem

$$\lambda(s) = e^{As} r(s) \tag{2.26}$$

where $r(s)$ is a rational function.

By (2.26), as $\sigma \to \infty$,

$$\lambda(\sigma) = C_0 \sigma^{m_0} e^{A\sigma}\big(1 + C_1\sigma^{-1} + O(\sigma^{-2})\big), \tag{2.27}$$

for some $C_0 \neq 0$ and $m_0 \in \mathbb{Z}$. On the other hand, if $b(n_0)$ is the first non-zero Dirichlet coefficient (with $n_0 > 1$) of $L_1(s)/L_2(s)$, then by (2.25) and Stirling's formula, as $\sigma \to \infty$,

$$\lambda(\sigma) = \big(B_0 \sigma^{B_1} e^{B_2 \sigma \log \sigma + B_3 \sigma}(1 + o(1))\big)\big(1 + b(n_0)n_0^{-\sigma} + O((n_0+1)^{-\sigma}\big). \tag{2.28}$$

Comparing those two asymptotic formulas, the leading terms must be equal, so $B_0 = C_0$, $B_1 = m_0$, $B_2 = 0$, and $B_3 = A$. Comparing second terms, we have polynomial decay equal to exponential decay, which is impossible unless $b(n_0) = 0$ and $C_1 = 0$. But $b(n_0)$ was the first nonzero coefficient of $L_1(s)/L_2(s)$, so we conclude that $L_1(s) = L_2(s)$, as claimed. □

The rest of this section is devoted to the proof of Lemma 2.6. By (2.8) and the partial Selberg bound assumed on μ and ν, only the product

$$P(s) = \prod_p \frac{F_{p,1}(p^{-s})^{-1}}{F_{p,2}(p^{-s})^{-1}} = \prod_p \frac{1 + a_1(p)p^{-s} + a_1(p^2)p^{-2s} + \cdots}{1 + a_2(p)p^{-s} + a_2(p^2)p^{-2s} + \cdots}$$

could contribute any zeros or poles to $\lambda(s)$ in the half-plane $\sigma \geq \frac{1}{2}$.

By the first line in equation (2.16) of Lemma 2.4 we have

$$\begin{aligned} P(s) &= \prod_p \frac{1 + a_1(p)p^{-s}}{1 + a_2(p)p^{-s}} \cdot \prod_p \frac{1 + a_1(p^2)p^{-2s}}{1 + a_2(p^2)p^{-2s}} \cdot H_1(s) \\ &= A_1(s)H_1(s), \end{aligned} \tag{2.29}$$

say, where $H_1(s)$ is regular and nonvanishing for $\sigma > \frac{1}{3} + \theta$.

Lemma 2.7. *Assuming $\theta < \frac{1}{6}$, bound* (2.9)*, and condition 1) in Theorem 2.2, with $A_1(s)$ as defined in* (2.29) *we have*

$$A_1(s) = \prod_p (1 + (a_1(p) - a_2(p))p^{-s}) \cdot \prod_p (1 + (a_1(p^2) - a_2(p^2))p^{-2s}) \cdot H_2(s), \tag{2.30}$$

where $H_2(s)$ is regular and nonvanishing for $\sigma > \frac{5}{12}$.

We finish the proof of Lemma 2.6 and then conclude with the proof of Lemma 2.7.

Using the notation of Lemma 2.7, write (2.30) as $A_1(s) = A_2(s)H_2(s)$. Since $A_1(s)$ and $H_2(s)$ are meromorphic in a neighborhood of $\sigma \geq \frac{1}{2}$, so is $A_2(s)$. Changing variables $s \mapsto s + \frac{1}{2}$, which divides the nth Dirichlet coefficient by $1/\sqrt{n}$, we can apply Lemma 2.5, using the estimate (2.9) and condition 1) to conclude that $A_2(s)$ has only finitely many zeros or poles in $\sigma \geq \frac{1}{2}$. Since the same is true of $H_1(s)$ and $H_2(s)$, we have shown that $P(s)$ has only finitely many zeros or poles in $\sigma \geq \frac{1}{2}$. This completes the proof for conditions 2a) and 1).

In the other cases, the proof is almost the same, using Lemma 2.4 to rewrite equation (2.29) in terms of $L_j^S(s, \mathrm{sym}^2)$ or $L_j^S(s, \mathrm{ext}^2)$, and using Lemma 2.5 for the factors that remain. This concludes the proof of Lemma 2.6. □

Proof of Lemma 2.7. Using the identities

$$\frac{1 + ax}{1 + bx} = 1 + (a - b)x - \frac{b(a - b)x^2}{1 + bx} \tag{2.31}$$

and

$$1 + ax + bx^2 = (1 + ax)\left(1 + \frac{bx^2}{1 + ax}\right) \tag{2.32}$$

we have

$$\frac{1 + ax}{1 + bx} = (1 + (a - b)x)\left(1 - \frac{b(a - b)x^2}{(1 + (a - b)x)(1 + bx)}\right). \tag{2.33}$$

Thus

$$\begin{aligned} \prod_p \frac{1 + a_1(p)p^{-s}}{1 + a_2(p)p^{-s}} &= \prod_p (1 + (a_1(p) - a_2(p))p^{-s}) \\ &\quad \times \prod_p \left(1 - \frac{a_2(p)(a_1(p) - a_2(p))p^{-2s}}{(1 + (a_1(p) - a_2(p))p^{-s})(1 + a_2(p)p^{-s})}\right) \\ &= \prod_p (1 + (a_1(p) - a_2(p))p^{-s}) \cdot h(s) \end{aligned} \tag{2.34}$$

say. We wish to apply Lemma 2.5 to show that $h(s)$ is regular and nonvanishing for $\sigma > \sigma_0$ for some $\sigma_0 < \frac{1}{2}$. Since $\theta < \frac{1}{6}$, if $\sigma \geq \frac{1}{6}$ and $p > P_0$ where P_0 depends only on θ, then $|1 + a_2(p)p^{-\sigma}| \geq \frac{1}{2}$ and $|1 + (a_1(p) - a_2(p))p^{-\sigma}| \geq \frac{1}{2}$. Using those inequalities and $|a_2(p)| \ll p^\theta$ we have

$$\begin{aligned}\sum_{P_0 \leq p \leq X} \left| \frac{a_2(p)(a_1(p) - a_2(p))}{(1 + (a_1(p) - a_2(p))p^{-\sigma})(1 + a_2(p)p^{-\sigma})} \right|^2 \log p \\ \leq 16 \sum_{P_0 \leq p \leq X} |a_2(p)(a_1(p) - a_2(p))|^2 \log p \\ \ll X^{2\theta} \sum_{P_0 \leq p \leq X} |(a_1(p) - a_2(p))|^2 \log p \\ \ll X^{\frac{1}{2}+2\theta}. \end{aligned} \tag{2.35}$$

Changing variables $s \to \frac{s}{2} - \frac{1}{12}$ and applying Lemma 2.5, we see that $h(s)$ is regular and nonvanishing for $\sigma > \frac{5}{12}$.

Applying the same reasoning to the second factor in (2.30) completes the proof. $\square$

2.5. **Proof of Lemma 2.5.** Two basic results which are used in this section are:

Lemma 2.8. *If* $\sum_{n \leq X} |a(n)| \ll X^{1+\epsilon}$ *for every* $\epsilon > 0$, *then* $\displaystyle\sum_{n=1}^\infty \frac{a(n)}{n^s}$ *converges absolutely for all* $\sigma > 1$.

Lemma 2.9. *If* $\sum_{n \leq X} |a(n)| \leq CX$ *as* $X \to \infty$, *then*

$$\sum_{n=1}^\infty \frac{a(n)}{n^\sigma} \leq \frac{C}{\sigma - 1} + O(1)$$

as $\sigma \to 1^+$.

Both of those results follow by partial summation.

We first state and prove a simplified version of Lemma 2.5.

Lemma 2.10. *Let*

$$L(s) = \prod_p \sum_{j=0}^\infty a(p^j)p^{-js} \tag{2.36}$$

and suppose there exists $M \geq 0$ *and* $\theta < \frac{1}{2}$ *so that* $|a(p^j)| \ll p^{j\theta}$ *and*

$$\sum_{p \leq X} |a(p)|^2 \log p \leq (1 + o(1))M^2 X. \tag{2.37}$$

Then $L(s)$ *is a nonvanishing analytic function in the half-plane* $\sigma > 1$. *Furthermore, if* $L(s)$ *has a meromorphic continuation to a neighborhood of* $\sigma \geq 1$, *then* $L(s)$ *has at most* M^2 *zeros or poles on the* $\sigma = 1$ *line.*

Note that, by the prime number theorem (2.10), the condition on $a(p)$ is satisfied if $|a(p)| \leq M$.

Proof. We have

$$\begin{aligned}
L(s) &= \prod_p \sum_{j=0}^{\infty} a(p^j)p^{-js} \\
&= \prod_p \Bigl(1 + a(p)p^{-s} + \sum_{j\geq 2} a(p^j)p^{-js}\Bigr) \\
&= \prod_p \bigl(1 + a(p)p^{-s}\bigr) \\
&\quad \times \prod_p \bigl(1 + a(p^2)p^{-2s} + (a(p^3) - a(p)a(p^2))p^{-3s} \\
&\qquad\qquad + (a(p^4) - a(p)a(p^3) + a(p)^2a(p^2))p^{-4s} + \cdots\bigr) \\
&= \prod_p \bigl(1 + a(p)p^{-s}\bigr) \prod_p \Bigl(1 + \sum_{j=2}^{\infty} b(p^j)p^{-js}\Bigr), \qquad (2.38)
\end{aligned}$$

say, where $b(p^j) \ll jM^jp^{j\theta} \ll p^{j(\theta+\epsilon)}$ for any $\epsilon > 0$.

Writing (2.38) as $L(s) = f(s)g(s)$ we have

$$\log g(s) = \sum_p \log(1+Y) = \sum_p \bigl(Y + O(Y^2)\bigr), \qquad (2.39)$$

where $Y = \sum_{j=2}^{\infty} b(p^j)p^{-js}$. Now,

$$\begin{aligned}
|Y| &\leq \sum_{j=2}^{\infty} |b(p^j)|p^{-j\sigma} \\
&\ll \sum_{j=2}^{\infty} p^{j(\theta-\sigma+\epsilon)} \\
&= \frac{p^{2(\theta-\sigma+\epsilon)}}{1 - p^{\theta-\sigma+\epsilon}}. \qquad (2.40)
\end{aligned}$$

If $\sigma > \frac{1}{2} + \theta$ we have $|Y| \ll 1/p^{1+\delta}$ for some $\delta > 0$. Therefore the series (2.39) for $\log(g(s))$ converges absolutely for $\sigma > \frac{1}{2} + \theta$, so $g(s)$ is a nonvanishing analytic function in that region. By (2.37), Cauchy's inequality, and Lemma 2.8, $f(s)$ is a nonvanishing analytic function for $\sigma > 1$, so the same is true for $L(s)$. This establishes the first assertion in the lemma.

Now we consider the zeros of $L(s)$ on $\sigma = 1$. Since $\theta < \frac{1}{2}$, the zeros or poles of $L(s)$ on the $\sigma = 1$ line are the zeros or poles of $f(s)$. Furthermore, by (2.38) and the properties of $g(s)$, for $\sigma > 1$ we have

$$\frac{L'}{L}(s) = \sum_p \frac{-a(p)\log(p)}{p^s} + h(s), \qquad (2.41)$$

where $h(s)$ is bounded in $\sigma > \frac{1}{2} + \theta + \epsilon$ for any $\epsilon > 0$. Suppose $s_1, \ldots, s_J$ are zeros or poles of $L(s)$, with $s_j = 1 + it_j$ having multiplicity m_j. We have

$$\frac{L'}{L}(\sigma + it_j) \sim \frac{m_j}{\sigma - 1}, \qquad \text{as} \quad \sigma \to 1^+, \qquad (2.42)$$

therefore

$$\sum_p \frac{-a(p)\log(p)}{p^{\sigma+it_j}} \sim \frac{m_j}{\sigma-1}, \qquad \text{as } \sigma \to 1^+. \tag{2.43}$$

Now write

$$k(s) = \sum_{j=1}^{J} m_j \sum_p \frac{-a(p)\log(p)}{p^{s+it_j}}. \tag{2.44}$$

By (2.43) we have

$$k(\sigma) \sim \frac{\sum_{j=1}^{J} m_j^2}{\sigma-1}, \qquad \text{as } \sigma \to 1^+. \tag{2.45}$$

On the other hand, for $\sigma > 1$ we have

$$\begin{aligned}
|k(\sigma)| &= \left| \sum_p \frac{a(p)\log(p)}{p^\sigma} \sum_{j=1}^{J} m_j p^{-it_j} \right| \\
&\le \left(\sum_p \frac{|a(p)|^2 \log(p)}{p^\sigma} \right)^{\frac12} \left(\sum_p \frac{\log p}{p^\sigma} \left| \sum_{j=1}^{J} m_j p^{-it_j} \right|^2 \right)^{\frac12} \\
&\le (1+o(1)) \left(\frac{M^2}{\sigma-1} \right)^{\frac12} \left(\sum_{j=1}^{J} \sum_{\ell=1}^{J} m_j m_\ell \sum_p \frac{\log p}{p^{\sigma+i(t_j-t_\ell)}} \right)^{\frac12} \\
&\sim \left(\frac{M^2}{\sigma-1} \right)^{\frac12} \left(\sum_{j=1}^{J} \frac{m_j^2}{\sigma-1} \right)^{\frac12} \qquad \text{as } \sigma \to 1^+.
\end{aligned} \tag{2.46}$$

On the first line we used the Cauchy-Schwartz inequality, on the next-to-last line we wrote the sum over $a(p)$ as a Stieltjes integral and used (2.37) and Lemma 2.9, and on the last line we used the fact that the Riemann zeta function has a simple pole at $s = 1$ and no other zeros or poles on the $\sigma = 1$ line.

Combining (2.43) and (2.46) we have $\sum_{j=1}^{J} m_j^2 \le M^2$. Since $m_j^2 \ge 1$, we see that $J \le M^2$, as claimed. □

The proof of Lemma 2.5 is similar to Lemma 2.10.

Proof of Lemma 2.5. We have

$$\begin{aligned}
L(s) &= \prod_p \sum_{j=0}^{\infty} a(p^j)p^{-js} \\
&= \prod_p \Big(1 + a(p)p^{-s} + a(p^2)p^{-2s} + \sum_{j\geq 3} a(p^j)p^{-js}\Big) \\
&= \prod_p \left(1 + a(p)p^{-s}\right)\left(1 + a(p^2)p^{-2s}\right) \\
&\qquad \times \big(1 + (a(p^3) - a(p)a(p^2))p^{-3s} \\
&\qquad\qquad + (a(p^4) - a(p)a(p^3) + a(p)^2 a(p^2))p^{-4s} + \cdots\big) \\
&= \prod_p \left(1 + a(p)p^{-s}\right)\left(1 + a(p^2)p^{-2s}\right) \prod_p \Big(1 + \sum_{j=3}^{\infty} c(p^j)p^{-js}\Big) \\
&= f(s)g(s), \qquad (2.47)
\end{aligned}$$

say.

We have $c(p^j) \ll jM^j p^{j\theta} \ll p^{j(\theta+\epsilon)}$ for any $\epsilon > 0$. We use this to show that $g(s)$ is a nonvanishing analytic function in $\sigma > \frac{1}{3} + \theta$. Writing $g(s) = \prod_p (1 + Y)$ we have

$$\log g(s) = \sum_p \log(1 + Y) = \sum_p \left(Y + O(Y^2)\right), \qquad (2.48)$$

where $Y = \sum_{j=3}^{\infty} b(p^j)p^{-js}$. Now,

$$\begin{aligned}
|Y| &\leq \sum_{j=3}^{\infty} |b(p^j)|p^{-j\sigma} \\
&\ll \sum_{j=3}^{\infty} p^{j(\theta-\sigma+\epsilon)} \\
&= \frac{p^{3(\theta-\sigma+\epsilon)}}{1 - p^{\theta-\sigma+\epsilon}}. \qquad (2.49)
\end{aligned}$$

If $\sigma > \frac{1}{3} + \theta$ we have $|Y| \ll 1/p^{1+\delta}$ for some $\delta > 0$. Therefore by Lemma 2.8 the series (2.48) for $\log(g(s))$ converges absolutely for $\sigma > \frac{1}{3} + \theta$, so $g(s)$ is a nonvanishing analytic function in that region. By the same argument, using (2.23) and (2.24), $f(s)$ is a nonvanishing analytic function for $\sigma > 1$, so the same is true for $L(s)$. This establishes the first assertion in the lemma.

Now we consider the zeros of $L(s)$ on $\sigma = 1$. Since $\theta < \frac{2}{3}$, the zeros or poles of $L(s)$ on the $\sigma = 1$ line are the zeros or poles of $f(s)$. Taking the logarithmic derivative of (2.47) and using the same argument as above for the lower order terms, we have

$$\begin{aligned}
\frac{L'}{L}(s) &= \sum_p \frac{-a(p)\log(p)}{p^s} + 2\,\frac{a(p)^2\log(p)}{p^{2s}} - 2\,\frac{a(p^2)\log(p)}{p^{2s}} + h_1(s) \\
&= \sum_p \frac{-a(p)\log(p)}{p^s} - 2\,\frac{a(p^2)\log(p)}{p^{2s}} + h_2(s), \qquad (2.50)
\end{aligned}$$

where $h_j(s)$ is bounded in $\sigma > \frac{1}{3} + \theta + \epsilon$ for any $\epsilon > 0$. By (2.23) and Lemma 2.8, the middle term in the sum over primes in (2.50) converges absolutely for $\sigma > \frac{1}{2}$, so it was incorporated into $h_1(s)$.

Suppose $s_1, \ldots, s_J$ are zeros or poles of $L(s)$, with $s_j = 1 + it_j$ having multiplicity m_j. We have

$$\frac{L'}{L}(\sigma + it_j) \sim \frac{m_j}{\sigma - 1}, \qquad \text{as } \sigma \to 1^+, \tag{2.51}$$

therefore

$$\sum_p \left(\frac{-a(p)\log(p)}{p^{\sigma+it_j}} - 2\,\frac{a(p^2)\log(p)}{p^{2(\sigma+it_j)}} \right) \sim \frac{m_j}{\sigma - 1}, \qquad \text{as } \sigma \to 1^+. \tag{2.52}$$

Now write

$$k(s) = \sum_{j=1}^{J} m_j \sum_p \left(\frac{-a(p)\log(p)}{p^{s+it_j}} - 2\,\frac{a(p^2)\log(p)}{p^{2(s+it_j)}} \right) \tag{2.53}$$

By (2.52) we have

$$k(\sigma) \sim \frac{\sum_{j=1}^{J} m_j^2}{\sigma - 1}, \qquad \text{as } \sigma \to 1^+. \tag{2.54}$$

We will manipulate (2.53) so that so that we can use (2.23) and (2.24) to give a bound on $\sum m_j^2$ in terms of M_1 and M_2.

By Cauchy's inequality and Lemma 2.9 we have

$$\begin{aligned}
|k(\sigma)| \le{}& \left| \sum_p \frac{a(p)\log(p)}{p^\sigma} \sum_{j=1}^{J} \frac{m_j}{p^{it_j}} \right| + 2 \left| \sum_p \frac{p^{-\sigma} a(p^2)\log(p)}{p^\sigma} \sum_{j=1}^{J} \frac{m_j}{p^{2it_j}} \right| \\
\le{}& \left(\sum_p \frac{|a(p)|^2 \log(p)}{p^\sigma} \right)^{\frac{1}{2}} \left(\sum_p \frac{\log p}{p^\sigma} \left| \sum_{j=1}^{J} m_j p^{-it_j} \right|^2 \right)^{\frac{1}{2}} \\
&+ 2 \left(\sum_p \frac{p^{-2\sigma} |a(p^2)|^2 \log(p)}{p^\sigma} \right)^{\frac{1}{2}} \left(\sum_p \frac{\log p}{p^\sigma} \left| \sum_{j=1}^{J} m_j p^{-2it_j} \right|^2 \right)^{\frac{1}{2}} \\
\le{}& (1 + o(1)) \Bigg(\left(\frac{M_1^2}{\sigma - 1} \right)^{\frac{1}{2}} \left(\sum_{j=1}^{J} \sum_{\ell=1}^{J} m_j m_\ell \sum_p \frac{\log p}{p^{\sigma + i(t_j - t_\ell)}} \right)^{\frac{1}{2}} \\
&+ 2 \left(\frac{M_2^2}{\sigma - 1} \right)^{\frac{1}{2}} \left(\sum_{j=1}^{J} \sum_{\ell=1}^{J} m_j m_\ell \sum_p \frac{\log p}{p^{\sigma + 2i(t_j - t_\ell)}} \right)^{\frac{1}{2}} \Bigg) \\
\sim{}& \frac{M_1 + 2M_2}{(\sigma - 1)^{\frac{1}{2}}} \left(\sum_{j=1}^{J} \frac{m_j^2}{\sigma - 1} \right)^{\frac{1}{2}} \qquad \text{as } \sigma \to 1^+. \qquad (2.55)
\end{aligned}$$

In the last step we used the fact that the Riemann zeta function has a simple pole at 1 and no other zeros or poles on the 1-line.

Combining (2.54) and (2.55) we have $\sum_{j=1}^{J} m_j^2 \le (M_1 + 2M_2)^2$. Since $m_j \ge 1$, the proof is complete. $\square$

3. Applications

3.1. **Strong multiplicity one for** $\mathrm{GL}(n)$**.** Let $\pi = \otimes\pi_p$ and $\pi' = \otimes\pi'_p$ be cuspidal automorphic representations of the group $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$. For a finite prime p for which π_p and π'_p are both unramified, let $A(\pi_p)$ (resp. $A(\pi'_p)$) represent the semisimple conjugacy class in $\mathrm{GL}(n, \mathbb{C})$ corresponding to π_p (resp. π'_p). The strong multiplicity one theorem for $\mathrm{GL}(n)$ states that if $A(\pi_p) = A(\pi'_p)$ for almost all p, then $\pi = \pi'$. The following result implies, in particular, that the equality of traces $\mathrm{tr}(A(\pi_p)) = \mathrm{tr}(A(\pi'_p))$ for almost all p is sufficient to reach the same conclusion. The traces could even be different at every prime, if those differences decreased sufficiently rapidly as a function of p.

Theorem 3.1. *Suppose that π and π' are (unitary) cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ satisfying*

$$\sum_{p \le X} p\,\log(p)\left|\mathrm{tr}A(\pi_p) - \mathrm{tr}A(\pi'_p)\right|^2 \ll X. \tag{3.1}$$

Assume a partial Ramanujan bound for some $\theta < \frac{1}{6}$ holds for both incomplete L-functions $L_{\mathrm{fin}}(s, \pi)$ and $L_{\mathrm{fin}}(s, \pi')$. Then $\pi = \pi'$.

Proof. We apply Theorem 2.2 to $L_1(s) = L_{\mathrm{fin}}(s, \pi)$ and $L_2(s) = L_{\mathrm{fin}}(s, \pi')$. The condition on the spectral parameters $\Re(\mu_j)$, $\Re(\nu_j) > -\frac{1}{2}$ is satisfied by Proposition 2.1 of [4]. By [7], the partial symmetric square L-function for $\mathrm{GL}(n)$ has meromorphic continuation to all of $\mathbb{C}$ and only finitely many poles in $\sigma \ge 1$. Using the fact that the partial Rankin-Selberg L-function of a representation of $\mathrm{GL}(n)$ with itself has no zeros in $\sigma \ge 1$ (see [25]) and that the partial exterior square L-function of $\mathrm{GL}(n)$ has only finitely many poles (see [5]), we see that partial symmetric square L-function for $\mathrm{GL}(n)$ has only finitely many zeros in $\sigma \ge 1$. This gives us condition 2b) of Theorem 2.2. The conclusion of Theorem 2.2 is that $L_1(s) = L_2(s)$. By the familiar strong multiplicity one theorem for $\mathrm{GL}(n)$, this implies $\pi_1 = \pi_2$. □

3.2. **Siegel modular forms.** In this section we prove Theorem 1.3. We start by giving some background on Siegel modular forms for $\mathrm{Sp}(4, \mathbb{Z})$. Let the symplectic group of similitudes of genus 2 be defined by

$$\mathrm{GSp}(4) := \{g \in \mathrm{GL}(4) : {}^tgJg = \lambda(g)J, \lambda(g) \in \mathrm{GL}(1)\} \qquad \text{where } J = \begin{bmatrix} & I_2 \\ -I_2 & \end{bmatrix}.$$

Let $\mathrm{Sp}(4)$ be the subgroup with $\lambda(g) = 1$. The group $\mathrm{GSp}^+(4, \mathbb{R}) := \{g \in \mathrm{GSp}(4, \mathbb{R}) : \lambda(g) > 0\}$ acts on the Siegel upper half space $\mathcal{H}_2 := \{Z \in M_2(\mathbb{C}) : {}^tZ = Z, \mathrm{Im}(Z) > 0\}$ by

$$g\langle Z\rangle := (AZ + B)(CZ + D)^{-1} \qquad \text{where } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GSp}^+(4, \mathbb{R}), Z \in \mathcal{H}_2. \tag{3.2}$$

Let us define the slash operator $|_k$ for a positive integer k acting on holomorphic functions F on $\mathcal{H}_2$ by

$$(F|_kg)(Z) := \lambda(g)^k \det(CZ + D)^{-k} F(g\langle Z\rangle), \qquad g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GSp}^+(4, \mathbb{R}), Z \in \mathcal{H}_2. \tag{3.3}$$

The slash operator is defined in such a way that the center of $\mathrm{GSp}^+(4,\mathbb{R})$ acts trivially. Let $S_k^{(2)}$ be the space of holomorphic Siegel cusp forms of weight k, genus 2 with respect to $\Gamma^{(2)} := \mathrm{Sp}(4,\mathbb{Z})$. Then $F \in S_k^{(2)}$ satisfies $F|_k\gamma = F$ for all $\gamma \in \Gamma^{(2)}$.

Let us now describe the Hecke operators acting on $S_k^{(2)}$. For a matrix $M \in \mathrm{GSp}^+(4,\mathbb{R}) \cap M_4(\mathbb{Z})$, we have a finite disjoint decomposition

$$\Gamma^{(2)} M \Gamma^{(2)} = \bigsqcup_i \Gamma^{(2)} M_i. \tag{3.4}$$

For $F \in S_k^{(2)}$, define

$$T_k(\Gamma^{(2)} M \Gamma^{(2)})F := \det(M)^{\frac{k-3}{2}} \sum_i F|_k M_i. \tag{3.5}$$

Note that this operator agrees with the one defined in [1]. Let $F \in S_k^{(2)}$ be a simultaneous eigenfunction for all the $T_k(\Gamma^{(2)} M \Gamma^{(2)}), M \in \mathrm{GSp}^+(4,\mathbb{R}) \cap M_4(\mathbb{Z})$, with corresponding eigenvalue $\mu_F(\Gamma^{(2)} M \Gamma^{(2)})$. For any prime number p, it is known that there are three complex numbers $\alpha_0^F(p), \alpha_1^F(p), \alpha_2^F(p)$ such that, for any M with $\lambda(M) = p^r$, we have

$$\mu_F(\Gamma^{(2)} M \Gamma^{(2)}) = \alpha_0^F(p)^r \sum_i \prod_{j=1}^{2} (\alpha_i^F(p) p^{-j})^{d_{ij}}, \tag{3.6}$$

where $\Gamma^{(2)} M \Gamma^{(2)} = \bigsqcup_i \Gamma^{(2)} M_i$, with

$$M_i = \begin{bmatrix} A_i & B_i \\ 0 & D_i \end{bmatrix} \quad \text{and} \quad D_i = \begin{bmatrix} p^{d_{i1}} & * \\ 0 & p^{d_{i2}} \end{bmatrix}. \tag{3.7}$$

Henceforth, if there is no confusion, we will omit the F and p in describing the $\alpha_i^F(p)$ to simplify the notations. The $\alpha_0, \alpha_1, \alpha_2$ are the classical Satake p-parameters of the eigenform F. It is known that they satisfy

$$\alpha_0^2 \alpha_1 \alpha_2 = p^{2k-3}. \tag{3.8}$$

For any $n > 0$, define the Hecke operators $T_k(n)$ by

$$T_k(n) = \sum_{\lambda(M)=n} T_k(\Gamma^{(2)} M \Gamma^{(2)}).$$

Let the eigenvalues of F corresponding to $T_k(n)$ be denoted by $\mu_F(n)$. Set $\alpha_p = p^{-(k-3/2)}\alpha_0$ and $\beta_p = p^{-(k-3/2)}\alpha_0\alpha_1$. Then formulas for the Hecke eigenvalues $\mu_F(p)$ and $\mu_F(p^2)$ in terms of α_p and β_p are

$$\mu_F(p) = p^{k-3/2}\big(\alpha_p + \alpha_p^{-1} + \beta_p + \beta_p^{-1}\big), \tag{3.9}$$

$$\mu_F(p^2) = p^{2k-3}\big(\alpha_p^2 + \alpha_p^{-2} + (\alpha_p + \alpha_p^{-1})(\beta_p + \beta_p^{-1}) + \beta_p^2 + \beta_p^{-2} + 2 - \frac{1}{p}\big). \tag{3.10}$$

The Ramanujan bound in this context is

$$|\alpha_p| = |\beta_p| = 1. \tag{3.11}$$

This is closely related to our use of that term for L-functions, as can be seen from the spin L-function of F:

$$L(s, F, \mathrm{spin}) = \prod_p F_p(p^{-s}, \mathrm{spin})^{-1},$$

where $F_p(X, \mathrm{spin}) = (1-\alpha_p X)(1-\alpha_p^{-1}X)(1-\beta_p X)(1-\beta_p^{-1}X)$. It satisfies the functional equation

$$\begin{aligned}\Lambda(s,F,\mathrm{spin}) &:= \Gamma_{\mathbb{C}}(s+\tfrac{1}{2})\Gamma_{\mathbb{C}}(s+k-\tfrac{3}{2})L(s,F,\mathrm{spin})\\ &= \varepsilon\Lambda(s,\overline{F},\mathrm{spin}),\end{aligned} \tag{3.12}$$

where $\varepsilon = (-1)^k$.

Let $a(p)$ be the pth Dirichlet coefficient of $L(s, F, \mathrm{spin})$. We will use the fact that each F falls into one of two classes.

i) $a(p) = p^{1/2} + p^{-1/2} + \beta_p + \beta_p^{-1}$, where β_p is the Satake p-parameter of a holomorphic cusp form on GL(2) of weight $2k-2$. In this case F is a *Saito-Kurokawa lifting*; for more details on Saito-Kurokawa liftings we refer to [9]. Note that $|\beta_p| = 1$, so that $a(p) = p^{1/2} + O(1)$ in the Saito-Kurokawa case.

ii) $a(p) = O(1)$. This is the Ramanujan conjecture for non-Saito-Kurokawa liftings, which has been proven in [27].

Theorem 1.3 is now a consequence of the following stronger result.

Theorem 3.2. *Suppose F_j, for $j = 1, 2$, are Siegel Hecke eigenforms of weight k_j for $\mathrm{Sp}(4,\mathbb{Z})$, with Hecke eigenvalues $\mu_j(n)$. If*

$$\sum_{p\le X} p\,\log(p)\left|p^{3/2-k_1}\mu_1(p) - p^{3/2-k_2}\mu_2(p)\right|^2 \ll X \tag{3.13}$$

as $X \to \infty$, then $k_1 = k_2$ and F_1 and F_2 have the same eigenvalues for the Hecke operator $T(n)$ for all n.

Proof. For $i = 1, 2$ let $a_i(p)$ be the pth Dirichlet coefficient of $L(s, F_i, \mathrm{spin})$. Then $a_i(p) = \alpha_{i,p} + \alpha_{i,p}^{-1} + \beta_{i,p} + \beta_{i,p}^{-1}$, where $\alpha_{i,p}, \beta_{i,p}$ are the Satake p-parameters of F_i, as explained after (3.8). By (3.9),

$$\mu_i(p) = p^{k_i-3/2}\big(\alpha_{i,p} + \alpha_{i,p}^{-1} + \beta_{i,p} + \beta_{i,p}^{-1}\big).$$

Hence, condition (3.13) translates into

$$\sum_{p\le X} p\,\log(p)\,|a_1(p) - a_2(p)|^2 \ll X. \tag{3.14}$$

From the remarks made before the theorem, we see that either F_1, F_2 are both Saito-Kurokawa lifts, or none of them is a Saito-Kurokawa lift.

Assume first that F_1, F_2 are both Saito-Kurokawa lifts. Then, for $i = 1, 2$, there exist modular forms f_i of weight $2k_i - 2$ and with Satake parameters $\beta_{i,p}$ such that $a_i(p) = p^{1/2} + p^{-1/2} + \beta_{i,p} + \beta_{i,p}^{-1}$. From (3.14) we obtain

$$\sum_{p\le X} p\,\log(p)\,|b_1(p) - b_2(p)|^2 \ll X, \tag{3.15}$$

where $b_{i,p} = \beta_{i,p} + \beta_{i,p}^{-1}$. Note that $b_{i,p}$ is the pth Dirichlet coefficient of (the analytically normalized L-function) $L(s, f_i)$. Since the Ramanujan conjecture is known for elliptic modular forms, Theorem 2.2 applies. We conclude $2k_1 - 2 = 2k_2 - 2$ and $L(s, f_1) = L(s, f_2)$. Hence $k_1 = k_2$ and $L(s, F_1, \mathrm{spin}) = L(s, F_2, \mathrm{spin})$. The equality of spin L-functions implies $\mu_1(p) = \mu_2(p)$ and $\mu_1(p^2) = \mu_2(p^2)$ for all p. Since $T(p)$ and $T(p^2)$ generate the p-component of the Hecke algebra, it follows that $\mu_1(n) = \mu_2(n)$ for all n.

Now assume that F_1 and F_2 are both not Saito-Kurokawa lifts. Then, using the fact that the Ramanujan conjecture is known for F_1 and F_2, Theorem 2.2 applies to $L_1(s) = L(s, F_1, \mathrm{spin})$ and $L_2(s) = L(s, F_2, \mathrm{spin})$. We conclude that $k_1 = k_2$ and that the two spin L-functions are identical. As above, this implies $\mu_1(n) = \mu_2(n)$ for all n. □

3.3. **Hyperelliptic curves.** Let $X/\mathbb{Q}$ be an elliptic or hyperelliptic curve,

$$X: \ y^2 = f(x),$$

where $f \in \mathbb{Z}[x]$, and let $N_X(p)$ be the number of points on X mod p. In Serre's recent book [24], the title of Section 6.3 is "About $N_X(p) - N_Y(p)$," in which he gives a description of what can happen if $N_X(p) - N_Y(p)$ is bounded. For Serre, X and Y are much more general than hyperelliptic cuves, but we use the hyperelliptic curve case to illustrate an application of multiplicity one results for L-functions.

Recall that the Hasse-Weil L-function of X,

$$L(X, s) = \sum_{n=1}^{\infty} \frac{a_X(n)}{n^s},$$

has coefficients $a_X(p^n) = p^n + 1 - N_X(p^n)$ if p is prime, which gives the general case by multiplicativity. The L-function (conjecturally if $g_X \geq 2$) satisfies the functional equation

$$\Lambda(X, s) = N_X^{s/2}\, \Gamma_{\mathbb{C}}(s)^{g_X} L(X, s) = \pm\Lambda(X, 2 - s), \tag{3.16}$$

where N_X is the conductor and $g_X = \lfloor(\deg(f) - 1)/2\rfloor$ is the genus of X.

Proposition 3.3. *Suppose X and Y are hyperelliptic curves and $N_X(p) - N_Y(p)$ is bounded. If the Hasse-Weil L-functions of X and Y satisfy their conjectured functional equation* (3.16)*, then X and Y have the same conductor and genus, and $N_X(p^e) = N_Y(p^e)$ for all p, e.*

Note that this result can be found in Serre's book without the hypothesis of functional equation. But Serre's proof involves more machinery than we use here.

Proof. To apply Theorem 2.2, we first form the *analytically normalized* L-function

$$L(s, X) = L(X, s + \tfrac{1}{2}) = \sum \frac{a_X(n)/\sqrt{n}}{n^s} = \sum \frac{b_X(n)}{n^s}, \tag{3.17}$$

say. Note that we have the functional equation

$$\Lambda(s, X) = N_X^{s/2}\Gamma_{\mathbb{C}}(s + \tfrac{1}{2})^{g_X} L(s, X) = \pm\Lambda(1 - s, X). \tag{3.18}$$

The Hasse bound for $a_X(n)$ implies the Ramanujan bound for $L(s, X)$. The condition $|N_X(p) - N_Y(p)| \ll 1$ is equivalent to

$$|b_X(p) - b_Y(p)| \ll \frac{1}{\sqrt{p}}, \tag{3.19}$$

which implies

$$\sum_{p \leq T} p|b_X(p) - b_Y(p)|^2 \log(p) \ll \sum_{p \leq T} \log(p) \sim T, \tag{3.20}$$

by the prime number theorem. Thus, Theorem 2.2 applies and we conclude that $L(X, s) = L(Y, s)$. □

If one knew that $L(s,X)$ and $L(s,Y)$ were "automorphic", then Theorem A.1 would apply, and a much weaker bound on $|N_X(p) - N_Y(p)|$ would allow one to conclude that $N_X(p^e) = N_Y(p^e)$ for all p, e. For example, if E, E' are elliptic curves over $\mathbb{Q}$, then $|N_E(p) - N_{E'}(p)| \le 1.4\sqrt{p}$ for all but finitely many p implies $N_E(p) = N_{E'}(p)$ for all p.

Appendix A. Selberg orthogonality and strong multiplicity one for GL(n)

The proof of Theorem 2.2 used only standard techniques from analytic number theory. Utilizing recent results concerning the Selberg orthonormality conjecture, and restrictng to the case of L-functions of cuspidal automorphic representations of $\mathrm{GL}(n)$, one obtains the following theorem, which is stronger than Theorem 3.1.

Theorem A.1. *Suppose that π, π' are (unitary) cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_F)$, and suppose*

$$\sum_{p \le X} \frac{1}{p} \left|\mathrm{tr}A(\pi_p) - \mathrm{tr}A(\pi'_p)\right|^2 \le (2-\epsilon)\log\log(X) \tag{A.1}$$

for some $\epsilon > 0$ as $X \to \infty$. If $n \le 4$, or if Hypothesis H holds for both $L_{\mathrm{fin}}(s,\pi)$ and $L_{\mathrm{fin}}(s,\pi')$ (in particular if the partial Ramanujan conjecture $\theta < \frac{1}{4}$ is true for π and π'), then $\pi = \pi'$.

Using the fact that $1.4^2 < 2$ and the consequence of the prime number theorem

$$\sum_{p \le X} \frac{1}{p} \sim \log\log(X), \tag{A.2}$$

we see that condition (A.1) holds if $\left|\mathrm{tr}A(\pi_p) - \mathrm{tr}A(\pi'_p)\right| < 1.4$ for all but finitely many p. Thus, the strong multiplicity one theorem only requires considering the traces of π_p, and futhermore those traces can differ at *every* prime, and by an amount which is bounded below.

For $GL(2, \mathbb{A}_{\mathbb{Q}})$, the Ramanujan bound along with (A.2) implies a version of a result of Ramakrishnan [17]: if $\mathrm{tr}A(\pi_p) = \mathrm{tr}A(\pi'_p)$ for $\frac{7}{8} + \varepsilon$ of all primes p, then $\pi = \pi'$. This result was extended by Rajan [18].

The proof of Theorem A.1 is a straightforward application of recent results toward the Selberg orthonormality conjecture [13, 3], which make use of progress on Rudnick and Sarnak's "Hypothesis H" [20, 11]. Suppose

$$L_1(s) = \sum \frac{a(n)}{n^s}, \qquad L_2(s) = \sum \frac{b(n)}{n^s} \tag{A.3}$$

are L-functions, meaning that they have a functional equation and Euler product as described in Section 2.1.

The point of the strong multiplicity one theorem is that two L-function must either be equal, or else they must be far apart. The essential idea was elegantly described by Selberg; see [23]. Recall that an L-function is *primitive* if it cannot be written nontrivially as a product of L-functions.

Conjecture A.2 (Selberg Orthonormality Conjecture)**.** *Suppose that L_1 and L_2 are primitive L-functions with Dirichlet coefficients $a(p)$ and $b(p)$. Then*

$$\sum_{p \le X} \frac{a(p)\overline{b(p)}}{p} = \delta(L_1, L_2)\log\log(X) + O(1), \tag{A.4}$$

where $\delta(L_1, L_2) = 1$ *if* $L_1 = L_2$*, and* 0 *otherwise.*

Proof of Theorem A.1. Rudnick and Sarnak's Hypothesis H is the assertion

$$\sum_p \frac{a(p^k)^2 \log^2(p)}{p^k} < \infty$$

for all $k \geq 2$. For a given k, this follows from a partial Ramanujan bound $\theta < \frac{1}{2} - \frac{1}{2k}$. Since $k \geq 2$, Hypothesis H follows from the partial Ramanujan bound $\theta < \frac{1}{4}$.

For the standard L-functions of cuspidal automorphic representations on $\mathrm{GL}(n)$, Rudnick and Sarnak [20] proved Selberg's orthonormality conjecture under the assumption of Hypothesis H, and they proved Hypothesis H for $n = 2, 3$. The case of $n = 4$ for Hypothesis H was proven by Kim [11]. Thus, under the conditions in Theorem A.1, the Selberg orthonormality conjecture is true.

Since π and π' are cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_F)$, the L-functions $L_1(s) = L_{\mathrm{fin}}(s, \pi)$ and $L_2(s) = L_{\mathrm{fin}}(s, \pi')$ are primitive L-functions. Hence, by (A.4)

$$\begin{aligned}\sum_{p \leq X} \frac{1}{p} |a(p) - b(p)|^2 &= \sum_{p \leq X} \frac{1}{p} \left(|a(p|^2 + |b(p)|^2 - 2\Re(a(p)\overline{b(p)}) \right) \\ &= 2 \log\log(X) - 2\delta_{L_1, L_2} \log\log(X) + O(1) \\ &= \begin{cases} O(1) & \text{if } L_1 = L_2 \\ 2\log\log(X) + O(1) & \text{if } L_1 \neq L_2. \end{cases} \end{aligned} \tag{A.5}$$

We have $\sum_{p \leq X} \frac{1}{p} |a(p) - b(p)|^2 \leq (2 - \epsilon) \log\log(X)$ for some $\epsilon > 0$. This implies that $\epsilon \log\log(X)$ is unbounded, and hence (A.5) implies that $L_1(s) = L_2(s)$. This gives us $\pi = \pi'$. □

Recently, the transfer of full level Siegel modular forms to $\mathrm{GL}(4)$ was obtained in [16]. Hence, we can apply Theorem A.1 to the transfer to $\mathrm{GL}(4)$ of a Siegel modular form of full level and thus obtain a stronger version of Theorem 3.2.

Theorem A.3. *Suppose F_j, for $j = 1, 2$, are Siegel Hecke eigenforms of weight k_j for $\mathrm{Sp}(4, \mathbb{Z})$, with Hecke eigenvalues $\mu_j(n)$. If*

$$\sum_{p \leq X} \frac{1}{p} \left| p^{3/2 - k_1} \mu_1(p) - p^{3/2 - k_2} \mu_2(p) \right|^2 \leq (2 - \epsilon) \log\log(X) \tag{A.6}$$

for some $\epsilon > 0$, as $X \to \infty$, then $k_1 = k_2$ and F_1 and F_2 have the same eigenvalues for the Hecke operator $T(n)$ for all n.

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