LIVING IT UP IN THE FORMAL WORLD: AN ABSTRACT ALGEBRAIST'S TEACHING JOURNEY

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Abstract algebra is a fascinating field of study among mathematics topics. Despite its importance, very little research has focused on the teaching of abstract algebra. In response to this deficiency, in this study we present an abstract algebra professor's daily activities and thought processes as shared through his teaching diaries with a team of two mathematics educators and another abstract algebraist over the period of two semesters. We examined how he was able to live in the formal world of mathematical thinking while also dealing with the many pedagogical challenges that were set before him during the lectures.

Keywords: Reflections on Teaching, Abstract Algebra, Formal World of Mathematical Thinking

Introduction

William Thurston (1994), the Fields medalist, posed the question: "How do mathematicians advance human understanding of mathematics?" In his view, "what we are doing is finding ways for people to understand and think about mathematics" (p.162). It is unclear to what extent this is manifested in teaching practices at the undergraduate level, however, which are largely undocumented in the literature. As Dreyfus (1991) suggested, "one place to look for ideas on how to find ways to improve students' understandings is the mind of the working mathematician. Not much has been written on how mathematicians actually work" (p. 29).

This statement is still relevant almost two decades later, as Speer, Smith, and Horvath (2010) believed that very little research has focused directly on teaching practice and what teachers do and think daily, in class and out, as they perform their teaching work. They found that often "strong content knowledge and the ability to structure it for students may be taken as sufficient for good teaching" (p. 111).

Research in pedagogy at the university level is fairly new, and regrettably the communication between the mathematicians and those outside of the community is often very limited. According to Byers (2007):

> People want to talk about mathematics but they don't. They don't know how. Perhaps they don't have the language, perhaps there are other reasons. Many mathematicians usually don't talk about mathematics because talking is not their thing – their thing is "doing" of mathematics. (p. 7)

In an attempt to close this gap, Hodgson (2012), in his plenary lecture at ICME 12, raised the point about the need for a community and forum where mathematicians and mathematics educators can work as closely as possible on teaching and learning mathematics. In recent years various institutes and individuals have been more willing to examine and reflect on their own teaching styles, leading to a growing body of research in this area. For example, a study by Paterson, Thomas and Taylor (2011) described a supportive and positive association of two groups of mathematicians and mathematics educators from the same university which allowed the "cross-fertilization of ideas" (p. 359). The group met on a regular basis and discussed teaching strategies while watching small clips of each other's videos during a teaching episode. Hannah, Stewart and Thomas (2011, 2013) indicated cases in which a mathematician took careful diaries of his/her actions and thoughts during linear algebra lectures and reflected on them with the rest of the team. Also, a study by Kensington-Miller, Yoon, Sneddon and Stewart (2013) showed how a mathematician, with the help of mathematics educators in a research team, made changes in his lecturing style while teaching a large undergraduate mathematics course by asking well-planned questions.

Naturally, the teacher and any corresponding methods of instruction do not stand alone in the classroom; they are subject to the needs and abilities of the students. In addition to a lack of information about teaching practices in abstract algebra, there is considerable evidence documenting student difficulty with the subject's most basic concepts (Clark, Hemenway, St. John, Tolias, & Vakil, 2007; Dubinsky, Dautermann, Leron, & Zazkis, 1994). This situation has led one group of researchers to starkly conclude that "the teaching of abstract algebra is a disaster" (Leron & Dubinsky, 1995, p. 227). To further investigate what makes this course so challenging, we will examine an abstract algebraist's daily mathematical activities through his teaching diaries to understand his way of thinking and possible challenges of teaching advanced mathematics courses that many mathematicians (and their students) may face. The overarching aim of this study is to investigate how mathematicians live and dwell in the formal world of mathematical thinking and, at the same time, communicate their knowledge to their students. Our research questions are: Given that the mathematician in this study is a formal thinker, how does he invite students to his world and to what extent is he willing to help students to reach the higher level of mathematical thinking?

In the next section we will explore a theoretical framework by Tall (2004, 2010, 2013) that is appropriate in guiding this research to help us understand more about mathematicians as formal thinkers.

Theoretical Framework

In his theory, Tall introduced a framework based on three worlds of mathematical thinking: the *conceptual embodiment*, *operational symbolism* and *axiomatic formalism*. The world of conceptual embodiment is based on "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognise properties and patterns...and other forms of figures and diagrams" (Tall, 2010, p. 22). Embodiment can also be perceived as the construction of complex ideas from sensory experiences, giving body to an abstract idea. The world of operational symbolism is the world of practicing sequences of actions which can be achieved effortlessly and accurately. The world of axiomatic formalism "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (p. 22). Tall (2013) suggested that:

> Formal mathematics is more powerful than the mathematics of embodiment and symbolism, which are constrained by the context in which the mathematics is used. Formal mathematics is future-proofed in the sense that any system met in the future that satisfies the definitions of a given axiomatic structure will also satisfy all the theorems proved in that structure. (p. 138)

In his view "research mathematicians will focus attention on the higher demands of research and assert professional standards appropriate at that level" (p. 143). As these levels are decidedly relevant in abstract algebra, we employed this framework as a means of differentiating and drawing comparisons between the varying levels of mathematical thinking exhibited by the mathematician in his journals.

Method

The research described here is a case study of a research mathematician that took place at a large research university in Fall 2012 and Spring 2013. The research team consisted of two mathematicians and two mathematics educators forming a community of enquiry. The data for this research comes from one of the research mathematicians' daily reflections on his teaching of an abstract algebra course, which were made available to the group after each class; the team members' observation of the classes and their comments; weekly discussion meetings of the whole group after reading each of these reflections and the audio recordings of each meeting which were later transcribed.

The mathematics professor in this study was an experienced faculty member who had taught many mathematics courses from college algebra to algebraic geometry. He captured many details in his daily diaries and shared them promptly with the rest of the research team. The journals were brief, often included technical language, and gave an impression to the reader of being present in the class. During the weekly meetings, the rest of the research team, having already read the journals, gave the mathematician an opportunity to discuss his teaching from the past week. This was followed by questions from the research team, which often generated additional discussions. Additionally, the mathematician welcomed unannounced visits to his class by other members of the team. During the course of the two semesters he planned and devised a few teaching experiments in his abstract algebra lectures. He was approachable and open to new ideas during the meetings, and even attended educational talks by graduate students in the mathematics department. His positive attitude toward teaching and education enabled the team to get as close as possible to his way of thinking and interacting in the classroom.

The main themes emerging from the data were: the role of questions during the lectures; the role of examples to preview and illustrate a concept; assessment of students, content, class as a whole and students' understanding; examining the content, textbook and homework; reflection on himself, teaching, preparation, interaction and content; teaching experiments, preparation, decision making before/during the class, philosophy, rapport, teaching observation based on experience and teaching details. The data related to teaching and reflection comprised half of the data. For the purpose of this paper we will only concentrate on teaching experience and reflections on teaching.

What we will illustrate next is a glimpse into the mathematician's daily activity as a formal thinker stepping out of the research world and entering into the classroom to teach.

Results

As mentioned above, the mathematician performed several teaching experiments during the course of two semesters. The teaching experiments often included more focus on students and less lecturing. In one occasion he wrote in his diary (April 1):

The second half of the class was spent on the notion of conjugacy classes. I did not do this at all the way I had prepared it. Somehow, the idea of me writing "Definition:..." etc seemed really boring. Instead, I decided to introduce the notion by means of the simplest non-trivial example, namely S_3. By now everyone is familiar with this group. I resorted to the trick I had used before of having not me, but a student write on the blackboard. I called for a volunteer; nobody was eager, but eventually one of the better students stood up. ...It is always interesting how the simple fact that it is not me but a student standing there seems to increase class participation immediately. I believe this example was very illustrative and they learned the notion of conjugacy class better than with a formal definition. I am now wondering whether to even do the formal definition at all next time we continue, or just leave it at that.

Here we see a formal thinker moving away from his comfort zone of definitions, theorems and proofs, reverting back to a simple example in the symbolic world of mathematical thinking. By involving a student and starting from an example he is attempting to reach out to his students and break down the abstraction of this concept. Suggesting that he viewed this break from the routine as successful, he mentioned that he is considering not making use of the formal definition at all the next time he teaches the course.

On another occasion (February 11) before the class he decided to invite students to construct their own proofs without writing it all on the board for them:

In this class we proved the main theorem of Galois theory. Looking at my notes right before class, I realized that all the pieces of the proof are in place, and there isn't really much more to do. So I decided to more or less have the students develop the proof. I started the class by not stating the theorem, but writing down some ingredients of the proof, without the students knowing that this is going to be a proof at all. Then I guided them towards the main theorem by asking questions. Within ten minutes the proof was complete, and only then did we state it formally as a theorem.

It was apparent that as an experienced mathematician he knew his material well but was consciously aware that his students were not yet at that level. This awareness of the discrepancy between his understanding and that of his students was a common theme throughout his journals. For example, on September 17, he wrote:

We spent the last 15 minutes proving that a field of fractions is a fraction field. I am afraid that the point of this was not entirely clear, and that it was in fact a little bit confusing. But I don't know how to do it better.

It was also clear that he was looking for ways to make the ideas that came so naturally to him more accessible to his students (February 13):

Then we formulated and proved a small Galois-theoretic result. This would have been kind of a boring afterthought to the main theorem, with no obvious immediate purpose. So I thought to myself right before class, how can I make this interesting? I resorted to the following trick: Before stating the theorem, I said that the level of complexity of the proof is such that a similar statement could easily be a problem on a qualifying exam. I think this kept everyone on their toes for the duration of the proof. I halfway had the class develop the proof, and it seemed like everyone was thinking hard, wanting to prove to themselves that they would be able to figure out a qualifying exam problem.

He knew that performing these teaching experiments would come with a cost, so he was consciously aware of the time and often was battling a tension between his identity as a mathematician (and the desire for conciseness and formality) with his identity as an instructor (and the desire to break down the material for his students) (October 22):

How did this happen? For one, I really wanted to get through with the proof of Gauss' theorem today. I knew time would be tight, and indeed we barely made it. So from the beginning I was in lecturing mode. Almost as if I didn't want the class to be disturbed by the possibility of students asking questions. This, of course, is a terrible attitude towards teaching.

As he reflected on his teaching, this conflict continued to be a challenge. After performing another teaching experiment he wrote: "While this was a very 'cool' and constructive class, we made zero progress on the material we are supposed to cover in this course" (November 2). Though teaching experiments comprised a very small portion of the course overall, they provided a rich source of insight into this mathematician's efforts to help his students navigate the formal nature of abstract algebra.

Although the professor was happy to try different teaching methods, at times he was not ready to change his beliefs on the usefulness of traditional lectures (October 15):

…I thought back about my own algebra education today, and how this was all very old-fashioned classroom lectures. There were not many questions asked by the teacher, and certainly there was never any group work or any kind of teaching experiment. Nevertheless, I remember thoroughly enjoying every class, with most of the fun coming from the beauty of the material itself. Made me wonder if it was just me having fun, or if there is more value in old-fashioned lecturing than we usually think there is.

The weekly discussions with the rest of the research team gave the mathematician another opportunity to reflect on his teaching and speak freely about the past week's events. It was noted that he was often excited about teaching, especially his favorite concepts and the materials that were well-prepared before the class (September 15):

> *…So when I prepare a class then it's also understanding for myself even though its stuff I know in principle but I put it fresh on my mind right, and it's almost like it wants to come out and it's really often that I wish the class would happen right now because I want to tell people about it now…*

Concluding Remarks

For a research mathematician, transitioning from the formal world of mathematical thinking back to the symbolic and embodied worlds is pedagogically challenging and requires an awareness of students' level of thinking and careful preparation. The results in this paper give a brief account of the mathematician's everyday actions and thoughts. Returning to our research questions, the results of this study provide some insight into the thought processes engaged in by a mathematician teaching an abstract algebra course. Specifically, this paper details his reflections on his efforts to help his students access the formal nature of abstract algebra. The results of this study provide a preliminary characterization of his efforts to do so. Of course, this is but a small portion of his journals and reflections (a full-scale report is beyond the scope of this proposal). The authors are in the process of making the full report of this research available in the near future.

Reflecting on the statement that "the teaching of abstract algebra is a disaster," the results of this two-semester study of an abstract algebra course suggest a positive outcome with regards to the collaboration between mathematicians and mathematics educators, despite the fact that 95% of the course consisted of old-fashioned blackboard lectures and no classroom technology was deployed. So far the effect of this collaboration has been positive in the sense that everyone in the group are not only focusing on the research mathematician's teaching strategies and thinking processes, but also their own teaching and decision making on a day-to-day basis. Moreover, it has provided a platform allowing mathematicians to talk about mathematics freely and share their pedagogical challenges with each other.

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