Recall that a well-ordered set is a simply ordered set \((A, <)\) for which every nonempty subset contains a smallest element. Every finite ordered set is well-ordered. Other examples of well-ordered sets are the set of positive integers \(\mathbb{Z}^+\) and the set \(\{1 - 1/n \mid n \in \mathbb{Z}^+\} \cup \{2 - 1/n \mid n \in \mathbb{Z}^+\}\) where both are endowed with the Archimedean ordering. On the other hand, neither \(\mathbb{Z}\) nor \(K = \{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\}\) nor \(\mathbb{R}^+ = (0, \infty)\) are well-ordered with the Archimedean ordering. Despite such examples Zermelo’s theorem (which is known to be equivalent to the axiom of choice) asserts that every set admits an ordering that is a well-ordering. See section 10 of Munkres book for a more detailed discussion of well-ordered sets and Zermelo’s theorem.

If \(A\) is a well-ordered set then \(A\) itself contains a smallest element which we will denote by \(a_0\). Moreover for each \(x \in A\), if it is nonempty the set \(\{y \in A \mid y > x\}\) must have a smallest element, and this smallest element is an immediate successor of \(x\). Thus every non-maximal element in a well-ordered set has an immediate successor. (However some elements in a well-ordered set may not have an immediate predecessor!) For each element \(x\) in a well ordered set \(A\) the section at \(x\) is defined to be the subset \(S_x = (\alpha, x) = [a_0, x) = \{y \in A \mid y < x\}\).

The uncountable ordinal space \(S_\Omega\) is an uncountable well-ordered set in which each section \(S_x\) is countable. This description of \(S_\Omega\) is justified by the following:

**Lemma 1.** There exists an uncountable well-ordered set \(A\) such that \(S_x\) is countable for each \(x \in A\), and any two uncountable well-ordered sets satisfying this property are order isomorphic (that is, they have the same order type).

**Proof.** We will take this lemma for granted with only some comments about the proof: The existence of such a well-ordered set is a simple consequence of Zermelo’s Theorem—see §10 of Munkres book for details. The uniqueness is usually proved by using ‘transfinite induction’.

The uncountable ordinal space \(S_\Omega\) is a subset of the closed uncountable ordinal space \(\overline{S}_\Omega\) which is defined by \(\overline{S}_\Omega = S_\Omega \cup \{\Omega\}\) with the well-ordering given by: (a) if \(x, y \in S_\Omega\) then \(x < y\) in \(\overline{S}_\Omega\) if \(x < y\) in \(S_\Omega\), and (b) if \(x \in S_\Omega\) then \(x < \Omega\). Notice that \(\Omega\) is a maximal element in \(\overline{S}_\Omega\) (but \(S_\Omega\) does not have a maximal element). The notation \(S_\Omega\) is justified by the observation that \(S_\Omega\) is the section of \(\Omega\) in \(\overline{S}_\Omega\). It is also not hard to see that the closure of \(S_\Omega\) in \(\overline{S}_\Omega\) is \(\overline{S}_\Omega\), which justifies the use of the bar notation. We also remark that \(S_\Omega\) is an uncountable well-ordered set but it does not satisfy the statement in the previous lemma because the section at \(\Omega\) is uncountable.

Recall that the collection of intervals with the forms (i) \([a_0, y), (ii) (x, y)\) and (iii) \((x, \Omega]\) where \(x, y \in S_\Omega\) form a basis for the order topology on \(\overline{S}_\Omega\), and that the intervals with forms (i) and (ii) comprise a basis for the order topology on \(S_\Omega\). One key property that is useful in describing the order topology on \(S_\Omega\) (and on \(\overline{S}_\Omega\)) is the following:

**Lemma 2.** Every countable subset of \(S_\Omega\) has an upper bound in \(S_\Omega\).
Proof. Let $B$ be a countable subset of $S_\Omega$. Then the set $\hat{B} = \bigcup_{b \in B} S_b$ is a countable set since it is a countable union of countable sets. As $S_\Omega$ is uncountable there must be an element $x \in S_\Omega - \hat{B}$. If $x < b$ for some $b \in B$ then $x \in S_b \subset \hat{B}$ which is a contradiction. Therefore $x$ is an upper bound for $B$.

As one illustration of the usefulness of lemma 2, we sketch a proof that $S_\Omega$ is sequentially compact: Suppose $(x_n)$ is a sequence in $S_\Omega$. Then $\{x_n \mid n \in \mathbb{Z}_+\}$ is a countable subset of $S_\Omega$ and hence it has an upper bound $z$ by lemma 2. The closed interval $[a_0, z]$ contains the sequence $(x_n)$ and this interval is compact by Theorem 27.1 of Munkres (note that $S_\Omega$ satisfies the least upper bound property since it is well-ordered). Therefore $(x_n)$ has a convergent subsequence in $[a_0, z]$ by Theorem 28.2. This shows that every sequence in $S_\Omega$ has a subsequence which converges in $S_\Omega$, and $S_\Omega$ is sequentially compact.

Now we will examine whether or not $S_\Omega$ satisfies the seventeen topological properties listed on page 228 of Munkres book, giving brief explanations in each case.

(1) $S_\Omega$ is not connected: Let $x$ be an element of $S_\Omega$ and let $x_1$ be its immediate successor. Then $[a_0, x_1)$ and $(x_1, +\infty)$ are disjoint open sets and they are nonempty because $x \in [a_0, x_1)$ and $x_1 \in (x_1, +\infty)$. So these two sets form a separation of $S_\Omega$, showing that $S_\Omega$ is not connected. (Note that neither $S_\Omega$ nor $S_\overline{\Omega}$ satisfy the second condition required to be a linear continuum, so Munkres Theorem 24.1 does not apply here.)

(2) $S_\Omega$ is not path connected: This follows from (1) since path connected spaces are connected.

(3) $S_\Omega$ is not locally connected: Let $x$ be an element of $S_\Omega$ which does not have an immediate predecessor. For example, $x$ could be chosen to be the smallest element of the set $\{y \in S_\Omega \mid S_y \text{ is infinite}\}$. Let $U$ be a neighborhood of $x$. Then $U$ must contain an open interval $I$ with $x \in I$, and $I$ must contain an element $y < x$ since otherwise the left endpoint of $I$ would be an immediate predecessor of $x$. Let $y_1$ be the immediate successor of $y$. Then $[a_0, y_1) \cap U$ and $(y_1, +\infty) \cap U$ are open sets in the subspace topology on $U$ which form a separation of $U$. (These sets are nonempty since $y \in [a_0, y_1) \cap U$ and $y_1 \in (y_1, +\infty) \cap U$.) Hence the neighborhood $U$ of $x$ cannot be connected. We conclude that $x$ has no connected neighborhoods, and $S_\Omega$ is not locally connected.

(4) $S_\Omega$ is not locally path connected: Locally path connected spaces are locally connected.

(5) $S_\Omega$ is not compact: Since compact spaces are Lindelöf this follows from (14) below.

(6) $S_\Omega$ is limit point compact: This argument is similar to our explanation above that $S_\Omega$ is sequentially compact. See also example 2 on page 179 of Munkres.

(7) $S_\Omega$ is locally compact Hausdorff: $S_\Omega$ is Hausdorff by (8). For each $x \in S_\Omega$ a basic neighborhood of $x$ is an interval, and this interval is contained in a closed interval which is compact since $S_\Omega$ satisfies the least upper bound property (Munkres Theorem 27.1). It follows that $S_\Omega$ is locally compact (definition on p. 182).

(8) $S_\Omega$ is Hausdorff: Simply ordered spaces are Hausdorff (Theorem 17.11).

(9) $S_\Omega$ is regular: This follows from (8) and (11) because $T_4$ spaces are always $T_3$.

(10) $S_\Omega$ is completely regular: This follows from (11) since normal spaces are completely regular by Urysohn’s lemma.

2
(11) $S_\Omega$ is normal: Every well-ordered space is normal (Theorem 32.4).

(12) $S_\Omega$ is first countable: Let $x$ be an element of $S_\Omega$ with immediate successor $x_1$. Then the collection of intervals $\{(t, x_1) \mid t \in S_x\}$ is a neighborhood base at $x$ and this neighborhood base is countable because $S_x$ is countable.

(13) $S_\Omega$ is not second countable: Second countable spaces are always Lindelöf (Theorem 30.3). So this result follows from (14). (It also follows from (15).)

(14) $S_\Omega$ is not Lindelöf: Let $U = \{S_y \mid y \in S_\Omega\}$. Each element of the collection $U$ is an open set, and, for each $x \in S_\Omega$, $x \in S_{x_1} \in U$ where $x_1$ is the immediate successor of $x$. Therefore $U$ is an open cover of $S_\Omega$. Suppose that $V = \{S_y \mid y \in B\}$ for some countable set $B \subset S_\Omega$. By lemma 2 above, $B$ has an upper bound $z \in S_\Omega$. But then $z \notin \bigcup V$ contradicting the supposition that $V$ is an open cover of $U$. This shows that $U$ has no countable subcover and that $S_\Omega$ is not Lindelöf.

(15) $S_\Omega$ is not separable: Let $B$ be a countable subset of $S_\Omega$. By lemma 2 $B$ has an upper bound $z$. So $B \subset [a_0, z]$ and $[a_0, z]$ is a closed set. Thus $\overline{B} \subset [a_0, z]$ which is a proper subset of $S_\Omega$ since $[a_0, z]$ is countable but $S_\Omega$ is uncountable. It follows that $B$ is not dense in $S_\Omega$ and that $S_\Omega$ has no countable dense subset.

(16) $S_\Omega$ is locally metrizable: Let $x$ be an element of $S_\Omega$ with immediate successor $x_1$. The open subset $[a_0, x_1] = [a_0, x] = S_{x_1}$ is countable. Therefore $[a_0, x]$ is second countable since the basic open intervals of forms (i), (ii) and (iii) described above form a countable family. We also know that $[a_0, x]$ is a compact Hausdorff space by theorems 17.11 and 27.1 (as described above). Thus $[a_0, x]$ is regular and second countable and this implies that it is metrizable by the Urysohn metrization theorem. Therefore every element of $S_\Omega$ has a metrizable neighborhood and $S_\Omega$ is locally metrizable (definition on p. 218 or p. 261).

(17) $S_\Omega$ is not metrizable: By (5) and (6), $S_\Omega$ is limit point compact but not compact. Since a metrizable limit point compact space is compact (Theorem 28.2) we conclude that $S_\Omega$ is not metrizable.

Next we discuss which of the 17 properties are satisfied by $S_\Omega$. None of the connectedness properties (1)–(4) hold for $S_\Omega$ for virtually the same reasons that they failed for $S_\Omega$, and $S_\Omega$ satisfies the separation axioms (8)–(11) just as $S_\Omega$ did. The remaining properties need to be examined separately for $S_\Omega$.

(1) $S_\Omega$ is not connected.

(2) $S_\Omega$ is not path connected.

(3) $S_\Omega$ is not locally connected.

(4) $S_\Omega$ is not locally path connected.

(5) $S_\Omega$ is compact: This follows from Theorem 27.1 since $S_\Omega$ is a closed interval $[a_0, \Omega]$ and $S_\Omega$ satisfies the least upper bound property.

(6) $S_\Omega$ is limit point compact: Compact spaces are limit point compact.
(7) $\overline{S}_\Omega$ is locally compact Hausdorff: Compact spaces are locally compact by definition.

(8) $\overline{S}_\Omega$ is Hausdorff.

(9) $\overline{S}_\Omega$ is regular.

(10) $\overline{S}_\Omega$ is completely regular.

(11) $\overline{S}_\Omega$ is normal.

(12) $\overline{S}_\Omega$ is not first countable: Every neighborhood of $\Omega$ in $\overline{S}_\Omega$ contains an interval $(x, \Omega]$ for some $x \in S_\Omega$ and that interval contains an element of $S_\Omega$ (in fact it contains the immediate successor of $x$). This implies that $\Omega$ is a limit point of $S_\Omega$ and that $\Omega$ is in the closure of $S_\Omega$. However if $(x_n)$ is a sequence in $S_\Omega$ then $(x_n)$ is contained in a closed interval $[a_0, z]$ for some $z \in S_\Omega$. (This was shown above in the proof that $S_\Omega$ is sequentially compact.) As a result, there is no sequence in $S_\Omega$ which converges to $\Omega$. If $S_\Omega$ was first countable then this would contradict the ‘sequence lemma’ Theorem 30.1(a). So $\overline{S}_\Omega$ cannot be first countable.

(13) $\overline{S}_\Omega$ is not second countable: Second countable spaces are first countable, so this follows from (12). It also follows from (15).

(14) $\overline{S}_\Omega$ is Lindelöf: Compact spaces are Lindelöf, so this follows from (5).

(15) $\overline{S}_\Omega$ is not separable: If $B$ is a countable dense subset of $\overline{S}_\Omega$ then $B \cap S_\Omega$ is a countable dense subset of $S_\Omega$ but this is impossible since we have shown above that $S_\Omega$ is not separable.

(16) $\overline{S}_\Omega$ is not locally metrizable: Locally metrizable spaces are always first countable (this is an easy exercise). So this result follows from (12).

(17) $\overline{S}_\Omega$ is not metrizable: This also follows from (12) since metrizable spaces are first countable.

It should also be mentioned that the fact that $S_\Omega$ is an open subspace of $\overline{S}_\Omega$ may allow for alternate explanations of some of the above. For example, we know that subspaces of metrizable spaces are always metrizable—therefore once $S_\Omega$ is shown not to be metrizable we can immediately conclude that $\overline{S}_\Omega$ is not metrizable as well. As another example, you might recall that open subspaces of separable spaces are separable—therefore we can argue that $\overline{S}_\Omega$ is not separable since its open subspace $S_\Omega$ is not separable. In general a topological property is said to be hereditary if every subspace of a space which satisfies the property also satisfies the property, and it is said to be hereditary for open subspaces if every open subspace of a space which satisfies the property also satisfies the property. Of the seventeen properties, which are hereditary and which are hereditary for open subspaces?