Instructions  Be sure to give thorough explanations of your work to guarantee getting full credit.

1. (20 points) Let \( f(x, y) = e^x \sin(x) \) and let \( P = (\pi, 1) \).
   (a) Compute all four second partials for \( f \).
   (b) Find the direction in which \( f \) increases most rapidly at \( P \).
   (c) Find the directional derivative for \( f \) at \( P \) in the direction of the vector \( 3i + 2j \).
   ANSWER: (a) \( f_{xx}(x, y) = -e^y \sin(x) \), \( f_{xy}(x, y) = f_{yx}(x, y) = e^y \cos(x) \) and \( f_{yy}(x, y) = e^y \sin(x) \).
   (b) The unit vector in the direction of \( \nabla f(x, y) = \langle -e, 0 \rangle \) is \( \langle -1, 0 \rangle = -\vec{i} \).
   (c) \( D_u f(\pi, 1) = -3e/\sqrt{13} \).

2. (15 points) Sketch the region in the \( xy \)-plane which is the domain of the function \( g(x, y) = \sqrt{y^2 - x^2} \).
   Then (in a separate picture) sketch 4 level curves for \( g \) to represent a contour plot for the function.
   ANSWER: I’ll leave it to you to draw the sketches but note that:
   (a) The domain of \( g \) is the set of all ordered pairs \( (x, y) \) for which \( y^2 - x^2 \geq 0 \). This consists of all points in the plane which either lie on or above both of the lines \( y = x \) and \( y = -x \), or lie on or below both of those lines.
   (b) The level curve at \( k \) for \( g(x, y) \) is the (1) empty set if \( k < 0 \), (2) the pair of intersecting lines \( y = \pm x \) if \( k = 0 \), or a (3) hyperbola with asymptotes \( y = \pm x \) if \( k > 0 \).

3. (15 points) Let \( g(x, y) = 6 - 3x^2 - y^2 \).
   (a) Find an equation for the plane tangent to the graph of \( g \) at the point \( (-1, 3, -6) \).
   (b) There is exactly one point on the graph of \( g \) whose tangent plane is parallel to the plane \( 6x + 4y + z = 0 \). Find that point.
   ANSWER: (a) Since \( \nabla g(-1, 3) = (6, -6) \), a normal vector for the tangent plane is \( (6, -6, -1) \). Since the tangent plane passes through \( (-1, 3, -6) \) it has equation \( 6(x + 1) - 6(y - 3) - 1(z + 6) = 0 \) which reduces to \( 6x - 6y - z + 18 = 0 \).
   (b) The point is \( (1, 2, -1) \). To find it note that we have to determine when the vectors \( (6, 4, 1) \) (which is a normal vector for the plane \( 6x + 4y + z = 0 \)) and \( (-6x, -2y, -1) \) are parallel.

4. (15 points) Find equations of (a) the tangent plane and (b) the normal line to the surface with equation \( xyz = 30 \) at the point \( (2, 3, 5) \).
   ANSWER: (a) The tangent plane has equation \( 15x + 10y + 6z - 90 = 0 \).
   (b) The normal line has scalar parametrization \( x = 15t + 2, y = 10t + 3, z = 6t + 5 \). It can also be described by the symmetric equations \( \frac{x - 2}{15} = \frac{y - 3}{10} = \frac{z - 5}{6} \).

5. (20 points) Let \( f(x, y) = x^2 - 2xy + \frac{1}{5}y^3 - 3y \). Find all critical points for \( f \) and classify each as local maximum, local minimum or saddle point.
   ANSWER: Every critical point of this function is a stationary point (that is, a point where the gradient of \( f(x, y) \) equals the zero vector). Setting the gradient equal to the zero vector results in the pair of equations \( 2x - 2y = 0, -2x + y^2 - 3 = 0 \). Solving simultaneously leads to two critical points: \( (x_0, y_0) = (-1, -1) \) and \( (x_0, y_0) = (3, 3) \). For this function it is not hard to compute
   \[
   D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2 = 4y - 4 .
   \]
   Since \( D(-1, -1) \) is negative, \( (-1, -1) \) is a saddle point for \( f(x, y) \) (in other words it fails to be a local extreme). On the other hand, \( D(3, 3) \) is positive as is \( f_{xx}(3, 3) = 2 \), and so \( (3, 3) \) is a local minimum.
6. (15 points) Use the chain rule to find the partials of $w$ with respect to $s$ and $t$ where

$$w = x^2 y + z, \quad x = s \ln(t), y = te^s, z = 1/t .$$

**ANSWER:** We first review how to remember the chain rule in this setting. If $w = f(x, y, z)$ is a function of three variables then its differential $dw = df$ would be

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz$$

and this can also be written as

$$df = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz .$$

So, if $x$, $y$ and $z$ are in turn functions of variables $s$ and $t$ then $w$ can be viewed as a function of $s$ and $t$, and the partial derivatives of $w$ with respect to $s$ and $t$ are determined by:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$

and similarly

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} .$$

In the given situation we have:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s} = 2xy \cdot \ln(t) + x^2 \cdot te^s + 1 \cdot 0 = ste^s \ln(t)^2 (2 + s)$$

and

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t} = 2xy \cdot s/t + x^2 \cdot e^s + 1 \cdot (-1/t^2) = 2s^2 \ln(t)e^s + s^2 e^s \ln(t)^2 - 1/t^2 .$$