Basic Concepts of Point Set Topology

Notes for OU course Math 4853

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1. Introduction.

The definitions of 'metric space' and 'topological space' were developed in the early 1900's, largely through the work of Maurice Frechet (for metric spaces) and Felix Hausdorff (for topological spaces). The main impetus for this work was to provide a framework in which to discuss continuous functions, with the goal of examining their attributes more thoroughly and extending the concept beyond the realm of calculus. At the time, these mathematicians were particularly interested in understanding and generalizing the Extreme Value Theorem and the Intermediate Value Theorem. Here are statements of these important theorems, which are well-known to us from calculus:

THEOREM 1.1 (Extreme Value Theorem). *Every continuous function* $f : [a, b] \to \mathbb{R}$ *achieves a maximum value and a minimum value.*

THEOREM 1.2 (Intermediate Value Theorem). *If* $f : [a, b] \to \mathbb{R}$ *is a continuous function and* y *is a real number between* $f(a)$ *and* $f(b)$ *then there is a real number* x *in the interval* $[a, b]$ *such that* $f(x) = y$.

In these statements we recall the following basic terminology. For real numbers a and b, $[a, b]$ denotes the **closed finite interval**

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\} \ .$$

If c is a real number in the interval $[a, b]$ such that $f(c) \geq f(x)$ for every $x \in [a, b]$ then $f(c)$ is a **maximum value for** f **on** $[a, b]$, and if d is a real number in the interval $[a, b]$ such that $f(d) \leq f(x)$ for every $x \in [a, b]$ then $f(d)$ is a **minimum value for** f **on** $[a, b]$. The function $f : [a, b] \to \mathbb{R}$ is **continuous** provided that for each $x_0 \in [a, b]$ and each $\epsilon > 0$ there is a positive real number δ such that if $x \in [a, b]$ and $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. As we proceed to develop the ideas of point set topology, we will review and examine these notations more thoroughly.

Notice that in the statement of these two theorems the fact that the domain of the function f is a closed finite interval $[a, b]$ is crucial. Here are two examples that should convince you that this hypothesis is needed in both statements.

EXAMPLE 1.3. Consider the function $f : (0, 2] \to \mathbb{R}$ given by $f(x) = 1/x$. From our calculus experience we know that f is a continuous function (for example f is a rational function whose denominator does not have any roots inside the interval $(0, 2]$). This function has a minimum value of $f(2) = 1/2$ but it has no maximum value because $\lim_{x\to 0^+} f(x) = +\infty$. So the conclusion of the Extreme Value Theorem fails, but of course the 'half-open interval' $(0, 2] = \{x \in \mathbb{R} \mid 0 < x \le 2\}$ is not a closed finite interval—so the example does not violate the statement of the Extreme Value Theorem. □

EXAMPLE 1.4. Consider the function $g : [-1, 0) \cup (0, 2] \to \mathbb{R}$ given by $g(x) = 1/x$ (whose domain is the set $[-1, 0) \cup (0, 2] = \{x \in \mathbb{R} \mid -1 \le x \le 2 \text{ and } x \ne 0\}$). Again from calculus experience we know that g is a continuous function. This function satisfies $g(-1) = -1$ and $g(2) = 1/2$ however there is no real number x with $g(x) = 0$ even though $g(-1) < 0 < g(2)$. So the conclusion of the Intermediate Value Theorem fails, but again the set $[-1, 0) \cup (0, 2]$ is not a closed finite interval so the example does not contradict the statement of the Intermediate Value Theorem. [Sketch the graphs of the functions in these two examples so that you can better visualize what goes wrong.] □

Mathematicians in the early twentieth century, such as Frechet and Hausdorff, came to realize that the validity of the Extreme Value Theorem and the Intermediate Value Theorem particularly relied on the fact that any closed finite interval $[a, b]$ satisfies two key 'topological properties' known as 'compactness' and 'connectedness'. Our goal in the point-set topology portion of this course is to introduce the language of topology to the extent that we can talk about continuous functions and the properties of compactness and connectedness, and to use these ideas to establish more general versions of the Extreme Value and Intermediate Value Theorems. We will end with a discussion of the difference in perspective between the Frechet and Hausdorff definitions, and how attempts to bridge these definitions contributed to the early development of the subject of point set topology.

2. Review of Set Theory.

A set A is a collection of objects or elements. In order to avoid certain paradoxes of set theory we will assume that an 'object' is always chosen from some universal set.

If x is an element and A is a set then '$x \in A$' means that x is an element of A, and '$x \notin A$' means x is not an element in A. Sets are usually described by either listing all of their elements or by using 'set-builder notation'.

EXAMPLE 2.1. The set A consisting of three elements labeled by 1, 2 and 3 might be described in one of the following three ways:

$$\begin{aligned} A &= \{1, 2, 3\} \\ &= \{x \mid x \text{ is a positive integer and } x \le 3\} \\ &= \{x \mid x \text{ is a positive integer and } 0 < x^2 < 10\} \end{aligned}$$

The first description lists the elements of the set while the last two descriptions illustrate set-builder notation. □

EXAMPLES. Here are some important special examples of sets that we will refer to frequently:

- The **empty set** $\emptyset$ is the set which contains no elements. This means that $x \notin \emptyset$ for every element x in the universal set.
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is the **set of positive integers**.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the **set of integers**.
- $\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$ is the **set of rational numbers**. Recall that every rational number can be expressed as m/n where the greatest common divisor of m and n equals 1—when this happens we say that m/n is in reduced form. (For example, the reduced form for the rational number $39/52$ would be $3/4$.)
- $\mathbb{R}$ is the **set of real numbers**.
- $\mathbb{R} - \mathbb{Q} = \{x \in \mathbb{R} \mid x \neq \mathbb{Q}\}$ is the **set of irrational numbers**.
- $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ is the set of positive real numbers.
- If a and b are real numbers then we can define various 'intervals from a to b':

$$
\begin{aligned}
[a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \text{ called a closed interval} \\
(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \text{ called a closed interval} \\
[a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
(a, \infty] &= \{x \in \mathbb{R} \mid a < x\} \\
(-\infty, b] &= \{x \in \mathbb{R} x \leq b\}
\end{aligned}
$$

Notice that the 'interval' $[a, b]$ is the empty set if $a > b$, and that $[a, a] = \{a\}$. □

DEFINITION 2.2. We list some of the most important basic definitions in set theory. In the following let A and B be sets.

1. A **is a subset of** B (written $A \subseteq B$) provided that if $x \in A$ then $x \in B$.
2. A **is equal to** B (written $A = B$) if and only if A is a subset of B and B is a subset of A. This is equivalent to saying that x is an element of A iff x is an element of B. (Notation: 'iff' is short for 'if and only if'.)
3. A **is a proper subset of** B (written $A \subsetneq B$ or $A \subset B$) if A is a subset of B but A is not equal to B.
4. The **union of** A **and** B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
5. The **intersection of** A **and** B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.
6. The **set difference of** A **and** B is the set $A - B = \{x \mid x \in A \text{ and } x \notin B\}$. Notice that typically $A - B$ and $B - A$ are two different sets.

Definition 2.3. In addition to defining the union and intersection of two sets we can define the union and intersection of any finite collection of sets as follows. Let n be a positive integer (that is, $n \in \mathbb{Z}^+$), and let $A_1, A_2, \ldots, A_n$ be sets. Then define

$$\begin{aligned} A_1 \cup A_2 \cup \cdots \cup A_n &= \{x \mid x \in A_i \text{ for some } i \in \{1, 2, \ldots n\}\} \\ A_1 \cap A_2 \cap \cdots \cap A_n &= \{x \mid x \in A_i \text{ for every } i \in \{1, 2, \ldots n\}\} \end{aligned}$$

With a little more background we can also define unions and intersections of arbitrary families of sets. Let J be a non-empty set, and let A_j be a set for each $j \in J$. In this situation we say that $\{A_j \mid j \in J\}$ is a **family (or collection) of sets indexed by** J, and we refer to J as the **index set** for this family. The union and intersection of this family are respectively defined by the following:

$$\begin{aligned} \bigcup_{j \in J} A_j &= \bigcup \{A_j \mid j \in J\} = \{x \mid x \in A_j \text{ for some } j \in J\} \\ \bigcap_{j \in J} A_j &= \bigcap \{A_j \mid j \in J\} = \{x \mid x \in A_j \text{ for every } j \in J\} \end{aligned}$$

In particular notice that a finite union $A_1 \cup A_2 \cup \cdots \cup A_n$ is the same as $\bigcup_{j \in J} A_j$ where $J = \{1, 2, \ldots, n\}$, and similarly for intersections. Here are two examples of infinite unions and intersections.

Example 2.4. For each $n \in \mathbb{Z}^+$ let $A_n = [0, 1/n]$ (which is a closed interval in $\mathbb{R}$). Then $\{A_n \mid n \in \mathbb{Z}^+\}$ is a family of sets indexed by the positive integers, with $A_1 = [0, 1]$, $A_2 = [0, 1/2]$, $A_3 = [0, 1/3]$ and so on. Notice that each set A_n in this family contains its successor A_{n+1}, when this happens we say the family is a *decreasing family of nested sets*. Here we have

$$\bigcup_{n \in \mathbb{Z}^+} A_n = \bigcup_{n \in \mathbb{Z}^+} [0, 1/n] = [0, 1]$$

and

$$\bigcap_{n \in \mathbb{Z}^+} A_n = \bigcap_{n \in \mathbb{Z}^+} [0, 1/n] = \{0\}.$$

Since the determination of infinite unions and/or intersections like these can sometimes be difficult, let's give a formal proof of the second of these statements. As is often the case, being able to write down a formal proof is tantamount to being able to systematically determine the result in the first place.

Proof that $\bigcap_{n \in \mathbb{Z}^+} A_n = \{0\}$: First suppose that x is an element of $\bigcap_{n \in \mathbb{Z}^+} A_n$. By definition of intersection this means that $x \in A_n = [0, 1/n]$ for every positive integer n. In particular, x is an element of $A_1 = [0, 1]$ and so we must have $x \geq 0$. Suppose first that $x > 0$. Then by the Archimedean Principle[1] there is a positive integer N such that $1/N < x$. It follows that $x \notin [0, 1/N] = A_N$ which contradicts our choice of x as being an element of $\bigcap_{n \in \mathbb{Z}^+} A_n$.

[1] The *Archimedean Principle* asserts that for every positive real number x there is a positive integer N such that $1/N < x$. This is equivalent to the statement that the limit of the sequence $(1/n)_{n \in \mathbb{Z}^+}$ is 0.

Thus our supposition that $x > 0$ must be false (using 'proof by contradiction'). Since we have already observed that $x \geq 0$ it must be that $x = 0$. So we have that $x \in \{0\}$, and since x was an arbitrary element of $\bigcap_{n\in\mathbb{Z}^+} A_n$, this shows that $\bigcap_{n\in\mathbb{Z}^+} A_n \subseteq \{0\}$ (by the definition of subset).

Next, let's suppose that $x \in \{0\}$. Then $x = 0$ (since $\{0\}$ is a set with only one element). For each positive integer n, $x = 0$ is an element of $A_n = [0, 1/n]$. Therefore $x \in \bigcap_{n\in\mathbb{Z}^+} A_n$ by the definition of intersection, and this shows that $\{0\}$ is a subset of $\bigcap_{n\in\mathbb{Z}^+} A_n$. We have shown that each of the sets $\{0\}$ and $\bigcap_{n\in\mathbb{Z}^+} A_n$ is a subset of the other, and therefore the two sets are equal by the definition of set equality. $\square$

[See if you can write down a formal proof of the first statement above, and of the two claims in the next example.] $\square$

EXAMPLE 2.5. For each $n \in \mathbb{Z}^+$ let $B_n = [0, 1 - 1/n]$ (which is a closed interval in $\mathbb{R}$). This is a family of sets indexed by the set of positive integers, with $B_1 = [0, 0] = \{0\}$, $B_2 = [0, 1/2]$, $B_3 = [0, 2/3]$ and so on. Notice that each set B_n in this family is a subset of its successor B_{n+1}, when this happens we say the family is an *increasing family of nested sets*. Here we have

$$\bigcup_{n\in\mathbb{Z}^+} B_n = \bigcup_{n\in\mathbb{Z}^+} [0, 1 - 1/n] = [0, 1)$$

and

$$\bigcap_{n\in\mathbb{Z}^+} B_n = \bigcap_{n\in\mathbb{Z}^+} [0, 1 - 1/n] = \{0\}.$$

$\square$

The basic definitions of set theory satisfy a variety of well-known laws. For example, the '*associative law for union*' says that for any three sets A, B and C, $(A \cup B) \cup C = A \cup (B \cup C)$. And the '*commutative law for intersection*' says that for all sets A and B, $A \cap B = B \cap A$. Of particular interest are *DeMorgan's Laws* which can be expressed as follows: Let A be a set and let $\{B_j \mid j \in J\}$ be a family of sets then

$$\begin{aligned} A - \bigcup_{j\in J} B_j &= \bigcap_{j\in J} (A - B_j) \\ A - \bigcap_{j\in J} B_j &= \bigcup_{j\in J} (A - B_j) \end{aligned}$$

We will not give a comprehensive list of all of the elementary laws of set theory here, but any book on set theory or discrete mathematics will contain such lists. In particular, the theory of sets can be completely described by certain lists of laws, one well-known such list is referred to as the 'Zermelo-Fraenkel Axioms for Set Theory'.

Another definition involving sets that is very important and useful is the Cartesian product. Let n be a positive integer and let $A_1, A_2, \ldots, A_n$ be a family of n sets. Then the **Cartesian product** of this family is the set

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for each } i \in \{1, 2, \ldots, n\}\}.$$

The formal symbol $(a_1, a_2, \ldots, a_n)$ is called an **ordered n-tuple**, and we refer to a_i as the **ith coordinate of the n-tuple**. Notice that in this definition we don't require that the sets A_i all be distinct from each other. In fact, in the extreme case where all of the A_i's are the same set A we obtain

$$A^n \;=\; A \times A \times \cdots \times A \;=\; \{(a_1, a_2, \ldots, a_n) \mid a_i \in A \text{ for each } i \in \{1, 2, \ldots, n\}\}.$$

For example, $\mathbb{R}^n$ denotes the set of all ordered n-tuples of real numbers. One can also define infinite Cartesian products but we will not need them in this course.

If X is a set then the **power set of** X is the set of all subsets of X

$$\mathcal{P}(X) \;=\; \{A \mid A \subseteq X\}.$$

(Some authors denote the power set $\mathcal{P}(X)$ by 2^X.) Note that the empty set $\emptyset$ and the set X itself are always elements of $\mathcal{P}(X)$. If X is a finite set with n elements then $\mathcal{P}(X)$ is a set with 2^n elements.

3. Open Sets in the Euclidean Plane.

A very familiar example of a Cartesian product is the set of ordered pairs of real numbers

$$\mathbb{R}^2 \;=\; \{(x_1, x_2) \mid x_1 \in \mathbb{R} \text{ and } x_2 \in \mathbb{R}\}$$

which can be identified with the set of points in a plane using Cartesian coordinates. Writing $x = (x_1, x_2)$ and $y = (y_1, y_2)$, the Pythagorean Theorem leads to the formula

$$d(x, y) \;=\; \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

measuring the distance between the two points x and y in the plane. This distance function allows us to examine the standard Euclidean geometry of the plane, and for this reason we refer to $\mathbb{R}^2$ in conjunction with the distance formula d as the **Euclidean plane**. Given $x = (x_1, x_2) \in \mathbb{R}^2$ and a positive real number $\epsilon > 0$ we define the **open disk of radius ϵ centered at** x to be the set

$$B(x, \epsilon) \;=\; \{y \in \mathbb{R}^2 \mid d(x, y) < \epsilon\}$$

(which is also sometimes called an **open ball** and denoted by $B_\epsilon(x)$). Observe that $B(x, \epsilon)$ coincides with the set of all points inside (but not on) the circle of radius ϵ centered at x. We say that a subset $U \subseteq \mathbb{R}^2$ of the Euclidean plane is an **open set** provided that for each element $x \in U$ there is a real number $\epsilon > 0$ so that $B(x, \epsilon) \subseteq U$. This can be loosely paraphrased by saying $U \subseteq \mathbb{R}^2$ *is an open set iff for each element $x \in U$ all nearby points are contained in U*. However note that the term "nearby" must be interpreted in relative terms (which is equivalent to observing that different values of ϵ may be required for different points $x \in U$).

EXAMPLE 3.1. The set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 3\}$, which consists of all points lying strictly to the right of the vertical line $x_1 = 3$ in the x_1x_2-plane, is an open set in $\mathbb{R}^2$. However the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 3\}$, which consists of points on or to the right of $x_1 = 3$ is not an open set. Also each open disk $B(x, \epsilon)$ can be shown to be an open set in the Euclidean plane.

The next theorem provides an important description of the open sets in $\mathbb{R}^2$.

THEOREM 3.2. *The collection of open sets in the Euclidean plane $\mathbb{R}^2$ satisfies the following properties:*

(1) The empty set is an open set.
(2) The entire plane $\mathbb{R}^2$ is an open set.
(3) If U_j is an open set for each $j \in J$ then $\bigcup_{j \in J} U_j$ is an open set.
(4) If U_1 and U_2 are open sets then $U_1 \cap U_2$ is an open set.

Proof. Part (1) follows immediately from our definition of open set since the empty set does not contain any elements. To prove (2): suppose $x \in \mathbb{R}^2$ then $B(x, \epsilon)$ is contained in $\mathbb{R}^2$ for any choice of $\epsilon > 0$. Now consider (3). Let U_j be an open set for each $j \in J$ and let x be an element of the union $\bigcup_{j \in J} U_j$. Then $x \in U_{j_0}$ for some $j_0 \in J$ (definition of union). Since U_{j_0} is an open set, there is a real number $\epsilon > 0$ so that $B(x, \epsilon) \subseteq U_{j_0}$. Since $U_{j_0} \subseteq \bigcup_{j \in J} U_j$, it follows that $B(x, \epsilon) \subseteq \bigcup_{j \in J} U_j$, and this shows that $\bigcup_{j \in J} U_j$ is an open set.

Finally, consider (4). Let U_1 and U_2 be open sets, and let $x \in U_1 \cap U_2$ (which means that $x \in U_1$ and $x \in U_2$). Then there are real numbers $\epsilon_1 > 0$ such that $B(x, \epsilon_1) \subseteq U_1$, and $\epsilon_2 > 0$ such that $B(x, \epsilon_2) \subseteq U_2$. Let ϵ be the smaller of the two numbers ϵ_1 and ϵ_2. Then $\epsilon > 0$,

$$B(x, \epsilon) \subseteq B(x, \epsilon_1) \subseteq U_1$$

and

$$B(x, \epsilon) \subseteq B(x, \epsilon_2) \subseteq U_2.$$

Therefore, $B(x, \epsilon) \subseteq U_1 \cap U_2$ and it follows that $U_1 \cap U_2$ is an open set. $\square$

Note that the key observation in the final paragraph of the proof of the Theorem is that if $\epsilon \leq \epsilon_1$ then $B(x, \epsilon) \subseteq B(x, \epsilon_1)$ which follows immediately from the definitions of $B(x, \epsilon)$ and $B(x, \epsilon_1)$. This Theorem suggests the following general definition which is the central focus for 'point set topology' (this is essentially Hausdorff's definition):

DEFINITION 3.3. Let X be a set and let $\mathcal{T}$ be a family of subsets of X which satisfies the following four axioms:

(T1) The empty set $\emptyset$ is an element of $\mathcal{T}$.
(T2) The set X is an element of $\mathcal{T}$.
(T3) If $U_j \in \mathcal{T}$ for every $j \in J$ then $\bigcup_{j \in J} U_j$ is an element of $\mathcal{T}$.
(T4) If U_1 and U_2 are elements of $\mathcal{T}$ then the intersection $U_1 \cap U_2$ is an element of $\mathcal{T}$.

then we say that **$\mathcal{T}$ is a topology on the set X.** Here the axiom (T3) is called **closure of $\mathcal{T}$ under arbitrary unions** and axiom (T4) is called **closure of $\mathcal{T}$ under pairwise intersections**.

COMMENTS:

(i) Notice that the definition of topology seems unbalanced in the sense that unions and intersections are treated differently—we only require the intersection of two elements of $\mathcal{T}$ to be in $\mathcal{T}$, but we require arbitrary unions of elements of $\mathcal{T}$ to be in $\mathcal{T}$. However it's really this imbalance which contributes to the definition leading to a rich and useful theory.

(ii) WARNING: Every set X that has at least two elements will have more than one different collection of subsets that form a topology. That is, one set X will generally have MANY different possible topologies on it. So to specify a topology we have to specify both the set X and the collection of subsets $\mathcal{T}$.

Example 3.4. By theorem 3.2, the collection

$$\mathcal{T}_{euclid} \;=\; \{U \subset \mathbb{R}^2 \mid U \text{ is an open set in } \mathbb{R}^2\}$$

forms a topology on the Euclidean plane. This is called the **Euclidean topology on** $\mathbb{R}^2$. Notice that for each positive integer n the subset $U_n \subseteq \mathbb{R}^2$ defined by

$$U_n \;=\; \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 3 - 1/n\}$$

is an open set in $\mathbb{R}^2$, however the intersection

$$\bigcap_{n \in \mathbb{Z}^+} U_n \;=\; \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 3\}$$

is not an open set. This shows that part (4) of Theorem 3.2 will not be true for arbitrary intersections, and justifies the imbalance mentioned in Comment (i) above.

To end this section, we will describe how the example of the Euclidean topology of the plane can be generalized to 'Euclidean n-space'.

Example 3.5. For any positive integer n, we can endow the set $\mathbb{R}^n$ of ordered n-tuples of real numbers with a distance function

$$d(x, y) \;=\; \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. The set $\mathbb{R}^n$ together with the distance formula d is referred to as **Euclidean n-space**. For each $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and each real number $\epsilon > 0$, let

$$B(x, \epsilon) = \{y \in \mathbb{R}^n \mid d(x, y) < \epsilon\}.$$

Then a subset $U \in \mathbb{R}^n$ is said to be an **open set in Euclidean n-space** provided that for each $x \in U$ there is a real number $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. With this definition theorem 3.2 extends easily to describe the collection of open sets in Euclidean n-space (just replace $\mathbb{R}^2$ with $\mathbb{R}^n$ in the statement and proof of that theorem). As a consequence, it follows that the collection $\mathcal{T}_{euclid}$ of open sets in Euclidean n-space forms a topology on $\mathbb{R}^n$. This is called the **Euclidean topology on** $\mathbb{R}^n$.

Example 3.6. When $n = 1$, Euclidean n-space is called the **Euclidean line** $\mathbb{R}^1 = \mathbb{R}$. Here the distance function is given by

$$d(x_1, y_1) \;=\; \sqrt{(x_1 - y_1)^2} = |x_1 - y_1| \;,$$

for $x_1, y_1 \in \mathbb{R}$, and the open disks are open intervals:

$$B(x, \epsilon) \;=\; \{y \in \mathbb{R} \mid |x - y| < \epsilon\} \;=\; (x - \epsilon, x + \epsilon) \;.$$

Thus a subset U of the Euclidean line $\mathbb{R}$ is an open set iff for each $x \in U$ there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. In other words, $U \subset \mathbb{R}$ is an open set iff for each $x \in U$ there is an $\epsilon > 0$ such that if y is a real number with $|x - y| < \epsilon$ then $y \in U$. This collection of open sets forms the Euclidean topology $\mathcal{T}_{euclid}$ on the real line $\mathbb{R}$.

4. Review of Functions.

Let X and Y be sets. A **function f from X to Y** is a rule that associates each element of X with exactly one element of Y. For each element $x \in X$, the unique element of Y associated with x is denoted by $f(x)$ and called the **image of x under f**. We write $f : X \to Y$ to signify that f is a function from X to Y.

Given a function $f : X \to T$, the set X is referred to as the **domain of f**, and the set Y is called the **codomain of f**. The set $\{y \in Y \mid y = f(x) \text{ for some } x \in X\}$ is called the **range of the function f**. Notice that by definition the range of a function is a subset of its co-domain.

A function $f : X \to Y$ is said to be **one–to–one** (or **injective**) provided that if x_1 and x_2 are elements of X and $f(x_1) = f(x_2)$ then $x_1 = x_2$. Thus a function is one–to–one iff each element in the range of the function is associated with only one element of the domain. A function $f : X \to Y$ is **onto** (or **surjective**) provided that for each $y \in Y$ there is at least one $x \in X$ such that $f(x) = y$. In other words, $f : X \to Y$ is onto iff the range of f equals the codomain of f. A function $f : X \to Y$ which is both one–to–one and onto is called a **one–to–one correspondence** (or a **bijection**).

If $f : X \to Y$ and $g : Y \to Z$ are functions (where the codomain of f equals the domain of g) then we can form the **composition of f with g** to be the function $g \circ f : X \to Z$ given by $g \circ f(x) = g(f(x))$ for each $x \in X$. Note that the domain of $g \circ f$ equals the domain of f, while the codomain of $g \circ f$ equals the codomain of g. If X is a set and A is a subset of X then there is function $i : A \to X$ called the **inclusion function from A to X** defined by $i(a) = a$ for each $a \in A$. The inclusion function from a set X to itself is called the **identity function on X**.

Definition 4.1 (Images and Inverse Images of Sets). Let $f : X \to Y$ be a function with domain X and codomain Y. If $A \subset X$ then the **image of A under f** is the set

$$f(A) \ = \ \{y \in Y \mid y = f(a) \text{ for some } a \in A\} \ .$$

If B is a subset of Y then the **inverse image of B under f** is the set

$$f^{-1}(B) \ = \ \{x \in X \mid f(x) \in B\} \ .$$

These definitions can be summarized by saying that each function $f : X \to Y$ induces two functions on power sets. The first from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ given by $A \mapsto f(A)$ for each $A \in \mathcal{P}(X)$, and the second from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ given by $B \mapsto f^{-1}(B)$ for each $B \in \mathcal{P}(Y)$.

Example 4.2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 1$ for all $x \in \mathbb{R}$. Let A be the half-open interval $A = (1, 3]$, which in this example is a subset of both the domain and codomain of f. Then

$$f(A) = f((1,3]) = \{y \in \mathbb{R} \mid y = x^2 + 1 \text{ for some } x \in (1,3]\} = \{x^2 + 1 \mid 1 < x \leq 3\} = (2, 10]$$

and

$$\begin{aligned} f^{-1}(A) &= f^{-1}((1,3]) = \{x \in \mathbb{R} \mid x^2 + 1 \in (1,3]\} = \{x \mid 0 < x^2 \leq 3\} \\ &= \{x \mid 0 < x \leq \sqrt{3} \text{ or } -\sqrt{3} \leq x < 0\} = [-\sqrt{3}, 0) \cup (0, \sqrt{3}]. \end{aligned}$$

EXAMPLE 4.3. Let X be a set and let A be a subset of X. If $i : A \to X$ is the inclusion function and $B \subset X$ then

$$i^{-1}(B) \;=\; \{a \in A \mid i(a) = a \in B\} \;=\; \{aA \mid a \in A \text{ and } a \in B\} \;=\; A \cap B\ .$$

5. Definition of Topology and Basic Terminology.

Let's start by repeating the definition of a topology on a set X:

DEFINITION 5.1. Let X be a set and let $\mathcal{T}$ be a family of subsets of $\mathcal{T}$ which satisfies the following four axioms:

(T1) The empty set $\emptyset$ is an element of $\mathcal{T}$.
(T2) The set X is an element of $\mathcal{T}$.
(T3) If $U_j \in \mathcal{T}$ for each $j \in J$ then $\bigcup_{j \in J} U_j$ is in $\mathcal{T}$.
(T4) If U_1 and U_2 are in $\mathcal{T}$ then $U_1 \cap U_2$ is in $\mathcal{T}$.

then we say that $\mathcal{T}$ **is a topology on the set** X**.** Here the axiom (T3) is referred to by saying $\mathcal{T}$ **is closed under arbitrary unions** and axiom (T4) is referred by saying $\mathcal{T}$ **is closed under pairwise intersection**.

A set X together with a topology $\mathcal{T}$ on that set is called a **topological space**, more formally we will refer to this topological space as $(X, \mathcal{T})$. If $(X, \mathcal{T})$ is a topological space then the elements of $\mathcal{T}$ are called **open sets in** X.

THEOREM 5.2 ($\mathcal{T}$ is closed under finite intersections.). *Let $\mathcal{T}$ be a topology on a set X, and let $U_1, U_2, \ldots, U_n$ be open sets in this topology where n is a positive integer. Then the intersection $U_1 \cap U_2 \cap \cdots \cap U_n$ is an open set.*

Proof. We prove this statement by induction on n. If $n = 1$ then the collection of open sets has only one set U_1 and the intersection is just U_1 which is an open set. This shows that the statement is true when $n = 1$. Now suppose the statement is true for a positive integer n (this is the *induction hypothesis*). To complete the proof by induction we need to show that the statement is true for $n + 1$. So suppose that $U_1, U_2, \ldots, U_{n+1}$ are open sets. Then we can write $U_1 \cap U_2 \cap \cdots \cap U_{n+1} = V \cap U_{n+1}$ where $V = U_1 \cap U_2 \cap \cdots \cap U_n$. Then V is an open set by the induction hypothesis, and $V \cap U_{n+1}$ is open by axiom (T4). This shows that the statement is true for $n + 1$ (that is, $U_1 \cap U_2 \cap \cdots \cap U_{n+1}$ is open) and completes the proof by induction. $\square$

If x is an element of X and U is an open set which contains x then we say that U **is a neighborhood of** x.

THEOREM 5.3. *A subset $V \subseteq X$ is an open set if and only if every element $x \in V$ has a neighborhood that is contained in V.*

Proof. ($\Rightarrow$) Suppose that V is an open set in X. Then for every element $x \in V$, V is a neighborhood of x and $V \subseteq V$.

($\Leftarrow$) Suppose that V is a subset of X and each element $x \in V$ has a neighborhood U_x such that $U_x \subseteq V$. Then $V = \bigcup_{x\in V} U_x$ and since each U_x is open then the union is open by axiom (T3). This shows that V is an open set and completes the proof. $\square$

[In the second part of the previous proof, carefully explain the equality $V = \bigcup_{x\in V} U_x$?]

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set X and $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then we say that $\mathcal{T}_1$ **is finer than** $\mathcal{T}_2$, or that $\mathcal{T}_2$ **is coarser than** $\mathcal{T}_1$. If $\mathcal{T}_2 \subsetneq \mathcal{T}_1$ then $\mathcal{T}_1$ **is strictly finer than** $\mathcal{T}_2$. If neither of the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ are finer than the other then we say that the two topologies are **incomparable**. We end this section by describing a number of examples of topological spaces.

EXAMPLE 5.4. For any set X let $\mathcal{T}_{discrete} = \mathcal{P}(X) = \{A \mid A \subseteq X\}$. Clearly axioms (T1) and (T2) hold since $\emptyset$ and X are subsets of X. The union and the intersection of any collection of subsets of X is again a subset of X and this shows that axioms (T3) and (T4) hold as well. Therefore $\mathcal{T}_{discrete}$ forms a topology on X and this is called the **discrete topology on** X. So in the discrete topology, every subset of X is an open set. Therefore $\mathcal{T}_{discrete}$ is the largest possible topology on X, which is to say that the discrete topology is finer than every topology on X.

EXAMPLE 5.5. For any set X let $\mathcal{T}_{trivial} = \{\emptyset, X\}$. It is easily verified that $\mathcal{T}_{trivial}$ is a topology on X and this is called the **trivial topology on** X. This topology contains only two open sets (assuming that X is a nonempty set). Therefore $\mathcal{T}_{trivial}$ is the smallest possible topology on X, which is to say that every topology on X is finer than the trivial topology.

EXAMPLE 5.6. Just a reminder that for each positive integer n, $\mathcal{T}_{euclid}$ is a topology on $\mathbb{R}^n$ called the **Euclidean topology**. In particular, taking $n = 1$ gives the Euclidean topology $\mathcal{T}_{euclid}$ on the real line $\mathbb{R}$, referred to briefly as the **Euclidean line**. Here

$$\mathcal{T}_{euclid} = \{U \subset \mathbb{R} \mid \text{ for each } x \in U \text{ there is } \epsilon > 0 \text{ such that } (x-\epsilon, x+\epsilon) \subseteq U\}.$$

With this definition it is not hard to show that every open interval in $\mathbb{R}$ is an open set in the Euclidean topology. [Be sure you can write out an explanation for this.] But note carefully that there are open sets in the Euclidean topology that are not open intervals. For example, the union of two or more disjoint intervals (such as $(-1,1) \cup (\sqrt{2}, \pi)$) will be an open set which is not an interval.

EXAMPLE 5.7. Consider the collection of subsets $\mathcal{T}_\ell$ of the real line $\mathbb{R}$ given by

$$\mathcal{T}_\ell \;=\; \{U \subseteq \mathbb{R} \mid \text{ for each } x \in U \text{ there is } \epsilon > 0 \text{ so that } [x, x+\epsilon) \subseteq U\}.$$

This set forms a topology $\mathcal{T}_\ell$ on $\mathbb{R}$ called the **lower limit topology on** $\mathbb{R}$. It is not hard to show that (1) each finite half-open interval of the form $[a, b)$ where $a < b$ is an open set in the lower limit topology but not an open set in the Euclidean topology, while (2) every open set in the Euclidean topology on $\mathbb{R}$ is open in the lower limit topology. Thus the lower limit topology on $\mathbb{R}$ is strictly finer than the Euclidean topology on $\mathbb{R}$ (that is $\mathcal{T}_{euclid} \subsetneq \mathcal{T}_\ell$).

Example 5.8. Let X be a set and let x_0 be an element of X. Define $\mathcal{T}$ to be the collection of subsets of X consisting of X itself and all subsets of X which do not contain x_0. Thus

$$\mathcal{T} \;=\; \{X\} \cup \{U \subseteq X \mid x_0 \notin U\} \;.$$

This forms a topology on X which is called the **excluded point topology**.

Example 5.9. Let X be any set and define

$$\mathcal{T}_{cofinite} = \{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ is a finite set }\} \;=\; \{\emptyset\} \cup \{X - F \mid \; F \text{ is a finite subset of } X\}.$$

Then it is not hard to show that $\mathcal{T}_{cofinite}$ forms a topology on X, called the **cofinite topology on** X. (We will explain this using closed sets in the next section.) Note that if the set X itself is finite then every subset of X will be finite, and so the cofinite topology on a finite set X is the same as the discrete topology on X.

6. Closed Sets and the Closure Operation.

Throughout this section let $(X, \mathcal{T})$ be a topological space. A subset $C \subseteq X$ is said to be a **closed set in** X provided that its complement $X - C$ is an open set (that is, $X - C \in \mathcal{T}$). Notice that taking complements in X defines a bijection between the collection of open sets in a topological space and the collection of closed sets. Basic properties of closed sets are described by the next theorem, whose proof depends principally on DeMorgan's Laws.

Theorem 6.1. *The collection $\mathcal{C}$ of all closed sets in X satisfies the properties*

(1) The empty set $\emptyset$ is an element of $\mathcal{C}$.
(2) The set X is an element of $\mathcal{C}$.
(3) If $U_j \in \mathcal{C}$ for each $j \in J$ then $\bigcap_{j\in J} U_j$ is in $\mathcal{C}$.
(4) If U_1 and U_2 are in $\mathcal{C}$ then $U_1 \cup U_2$ is in $\mathcal{C}$.

Moreover, for any collection $\mathcal{C}$ of subsets of X which satisfies properties (1)–(4) there is a unique topology on X for which $\mathcal{C}$ is the collection of closed sets. □

Example 6.2. For any set X consider the collection of subsets given by

$$\mathcal{C} \;=\; \{X\} \cup \{F \subseteq X \mid F \text{ is a finite set }\} \;.$$

Then it is easy to check that $\mathcal{C}$ satisfies the four conditions in theorem 6.1 (for example, the intersection of any collection of finite sets is finite, and the union of two finite sets is finite). Therefore $\mathcal{C}$ forms the collection of closed sets for some topology on X, and this is the cofinite topology

$$\mathcal{T}_{cofinite} \;=\; \{\emptyset\} \cup \{U \subseteq X \mid X - U \text{ is a finite set }\}$$

as described in example 5.9.

If A is an arbitrary subset of X then the **closure of** A is the intersection $cl(A)$ of all closed sets in X which contain A. The next theorem is easily proved using this definition. The first two parts of this theorem can be summarized by saying that *the closure* $cl(A)$ *of a subset* A *is the smallest closed set which contains* A.

THEOREM 6.3. *For any subset* $A \subseteq X$ *the closure* $cl(A)$ *is a closed set. If* C *is a closed set containing* A *then* $cl(A) \subseteq C$. *The subset* A *is a closed set if and only if* $cl(A) = A$. □

EXAMPLE 6.4. Consider the trivial topology $\mathcal{T}_{trivial} = \{\emptyset, X\}$ on a set X. The closed sets in this topology are $X - \emptyset = X$ and $X - X = \emptyset$. Then the closure of a subset A of X is described by

$$cl(A) \ = \ \begin{cases} \emptyset, \text{ if } A = \emptyset \\ X, \text{ if } A \neq \emptyset. \end{cases}$$

To explain: If A is the empty set then both closed sets $\emptyset$ and X contain A, so $cl(A) = \emptyset \cap X = \emptyset$. If A is a nonempty subset of X then the only closed set containing A is X and so the intersection of all the closed sets containing A is X.

EXAMPLE 6.5. Consider the discrete topology $\mathcal{T}_{discrete} = \mathcal{P}(X)$ on a set X. Then every subset of X is closed, and so here the closure operation is described by $cl(A) = A$ (by theorem 6.3).

DEFINITION 6.6. Let A be a subset of X. An element $x \in X$ is called a **limit point of** A provided that every neighborhood U of x contains an element of A other than x itself. The set of all limit points of A is called the **derived set of** A and denoted by A'.

The next theorem describes a relationship between the set of limit points of a set and the closure of a set. The proof of this theorem is quite typical of proofs in point set topology.

THEOREM 6.7. *For any subset* $A \subseteq X$, $cl(A) = A \cup A'$.

Proof. The proof is broken into two parts where first we will show that $cl(A) \subseteq A \cup A'$ and then that $A \cup A' \subseteq cl(A)$.

$cl(A) \subseteq A \cup A'$: Suppose $x \in cl(A)$. Then we must show that x is an element of $A \cup A'$. For contradiction, let us assume that $x \notin A \cup A'$. Then $x \notin A$ and $x \notin A'$ (by the definition of union). Since $x \notin A'$, x is not a limit point of A and this means that there is a neighborhood U of x which contains no element of A other than x. In fact, because $x \notin A$, U cannot contain any elements of A. Thus A is a subset of $X - U$. Since $X - U$ is a closed set (as U is open) and $cl(A)$ is the smallest closed set containing A (by theorem 6.3), $cl(A)$ must be a subset of $X - U$. On the other hand, x is an element of $cl(A)$ (by supposition) but $x \notin X - U$ (since U is a neighborhood of x). This yields a contradiction and shows that $x \in A \cup A'$, as desired.

$A \cup A' \subseteq cl(A)$: Suppose $x \in A \cup A'$. For contradiction, let's assume that $x \notin cl(A)$. This means that $x \notin A$ (since A is a subset of $cl(A)$) and that $x \in A'$ (because $x \in (A \cup A') - A$). Observe that $X - cl(A)$ is an open set (being the complement of the closed set $cl(A)$), and that $X - cl(A)$ is a neighborhood of x ($x \in X - cl(A)$ since $x \notin cl(A)$). As x is a limit point of A, this neighborhood $X - cl(A)$ must contain an element of A. But since A is a subset of

$cl(A)$ it can't contain any elements of $X - cl(A)$. This contradiction leads us to conclude that $x \in cl(A)$, showing that $A \cup A' \subseteq cl(A)$. $\square$

Theorem 6.7 is often helpful for determining the closure $cl(A)$ of a given subset A in a topological space. However, it is not so useful for determining the derived set A', in part this is because there is no general relationship between the sets A and A' (in particular, these two sets need not be disjoint).

EXAMPLE 6.8. In the Euclidean line $(\mathbb{R}, \mathcal{T}_{euclid})$ consider the subsets

$$A_1 = (0,3), \quad A_2 = (0,3) \cup \{5\} \quad \text{and} \quad A_3 = \{1/n \mid n \in \mathbb{Z}^+\} .$$

Then it is not difficult to determine the derived sets to be

$$A_1' = [0,3], \quad A_2' = [0,3] \quad \text{and} \quad A_3' = \{0\} ,$$

and from this we find the closures

$$cl(A_1) = [0,3], \quad cl(A_2) = [0,3] \cup \{5\} \quad \text{and} \quad cl(A_3) = \{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\} .$$

Notice that of these three sets A it's only the last one where $A \cap A' = \emptyset$.

7. Continuous Functions and Subspaces.

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A function $f : X \to Y$ is said to be **continuous** provided that $f^{-1}(U) \in \mathcal{T}_1$ whenever $U \in \mathcal{T}_2$. To summarize this situation we will sometimes write that $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is a continuous function.

EXAMPLE 7.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$, and the lower limit topology $\mathcal{T}_\ell$ on $\mathbb{R}$. The half-open interval $[2,3)$ is open in the lower limit topology but

$$f^{-1}([2,\infty)) = \{x \in \mathbb{R} \mid x^2 \in [2,\infty)\} = \{x \in \mathbb{R} \mid 2 \le x^2\} = (-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$$

is not in $\mathcal{T}_\ell$ (because $-\sqrt{2} \in f^{-1}([2,\infty))$ but $[-\sqrt{2}, -\sqrt{2}+\epsilon)$ is not a subset of $f^{-1}([2,\infty))$ for any $\epsilon > 0$.) This shows that $f : (\mathbb{R}, \mathcal{T}_\ell) \to (\mathbb{R}, \mathcal{T}_\ell)$ is not continuous. Notice that since $f^{-1}([2,3))$ is not open in the Euclidean topology, this also shows that $f : (\mathbb{R}, \mathcal{T}_{euclid}) \to (\mathbb{R}, \mathcal{T}_\ell)$ is not continuous. But $f : (\mathbb{R}, \mathcal{T}_{euclid}) \to (\mathbb{R}, \mathcal{T}_{euclid})$ is continuous (see theorem 7.4 below), and $f : (\mathbb{R}, \mathcal{T}_\ell) \to (\mathbb{R}, \mathcal{T}_{euclid})$ is also continuous.

THEOREM 7.2. *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is a closed set in X whenever C is a closed set in Y.*

Proof. Suppose that $f : X \to Y$ is continuous and that C is a closed set in Y. Then $Y - C$ is an open set in Y, and so $f^{-1}(Y - C)$ is an open set in X by the definition of continuity. Then

$$\begin{aligned} X - f^{-1}(Y - C) &= \{x \in X \mid x \notin f^{-1}(Y - C)\} = \{x \in X \mid f(x) \notin Y - C\} \\ &= \{x \in X \mid f(x) \in C\} = f^{-1}(C) . \end{aligned}$$

Since $X - f^{-1}(Y - C)$ is a closed set (because $f^{-1}(Y - C)$ is open), this shows that $f^{-1}(C)$ is a closed set in X. The other direction of the if-and-only-if statement is proved in a similar fashion. □

EXAMPLE 7.3. Again consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ but this time with respect to the cofinite topology $\mathcal{T}_{cofinite}$ on $\mathbb{R}$. In this topology a closed set is either $\mathbb{R}$ or a finite subset $F \subset \mathbb{R}$. Since $f^{-1}(\mathbb{R}) = \mathbb{R}$ and $f^{-1}(F)$ is finite [Can you explain this? How many elements does $f^{-1}(F)$ have?], the function $f : (\mathbb{R}, \mathcal{T}_{cofinite}) \to (\mathbb{R}, \mathcal{T}_{cofinite})$ is continuous by the previous theorem.

THEOREM 7.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f : (\mathbb{R}, \mathcal{T}_{euclid}) \to (\mathbb{R}, \mathcal{T}_{euclid})$ is continuous if and only if $f : \mathbb{R} \to \mathbb{R}$ is continuous with the $\epsilon\delta$–definition of continuity.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Before starting on the proof let's recall that the $\epsilon\delta$–definition of continuity for f means: for each $x_0 \in \mathbb{R}$ and each $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

First assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous with the $\epsilon\delta$–definition. Suppose that $U \in \mathcal{T}_{euclid}$ and that $x_0 \in f^{-1}(U)$. By the definition of the Euclidean topology on $\mathbb{R}$, there is an $\epsilon > 0$ such that $B(f(x_0, \epsilon)) = (f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq U$. By the $\epsilon\delta$–definition, there is a $\delta > 0$ so that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. Thus if $|x - x_0| < \delta$ then $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$. It follows that

$$f^{-1}(U) \supseteq f^{-1}((f(x_0) - \epsilon, f(x_0) + \epsilon)) \supseteq \{x \in \mathbb{R} \mid |x - x_0| < \delta\} = (x_0 - \delta, x_0 + \delta) \ .$$

So we have shown that if $x_0 \in f^{-1}(U)$ then there exists $\delta > 0$ with $B(x_0, \delta) \subseteq f^{-1}(U)$ and this implies that $f^{-1}(U)$ is open in the Euclidean topology on $\mathbb{R}$. Since U was an arbitrary open set we have established that f is continuous.

Now assume that f is continuous (with the topology definition). Let $x_0 \in \mathbb{R}$ and let ϵ be a positive real number. The open interval $U = (f(x_0) - \epsilon, f(x_0) + \epsilon)$ is an open set in the Euclidean topology. Therefore $f^{-1}(U)$ is an open set as well (applying the continuity of f). Since $x_0 \in f^{-1}(U)$ (because $f(x_0) \in U$) there is positive real number δ with $B(x_0, \delta) \subseteq f^{-1}(U)$. If x is a real number with $|x - x_0| < \delta$ then $x \in B(x_0, \delta)$ and therefore $x \in f^{-1}(U)$. This means that $f(x)$ is an element of $U = (f(x_0) - \epsilon, f(x_0) + \epsilon)$, in other words $|f(x) - f(x_0)| < \epsilon$. Thus we have shown that f is continuous with the $\epsilon\delta$–definition of continuity, and that completes the proof. □

CONSEQUENCE: Every function $f : \mathbb{R} \to \mathbb{R}$ known to be continuous in Calculus I is continuous with respect to the Euclidean topology on $\mathbb{R}$. For example, polynomial functions are continuous as is $f(x) = \cos(x)$, and etc. This consequence even extends to functions $f : X \to \mathbb{R}$ where X is a subset of $\mathbb{R}$ (here we take the topology on X to be the 'subspace topology' coming from the Euclidean topology on $\mathbb{R}$ as described below). Thus for example, rational functions and the natural logarithm function are continuous when we take the domain of the function to be the 'domain of definition' (that is, the set of all real numbers for which the equation makes sense).

Let $(X, \mathcal{T})$ be a topological space, and let A be a subset of X. We define a collection $\mathcal{T}_A$ of subsets of A by:

$$\mathcal{T}_A \ = \ \{V \subset A \mid V = U \cap A \text{ for some } U \in \mathcal{T}\} \ = \ \{U \cap A \mid U \in \mathcal{T}\} \ .$$

THEOREM 7.5. *Let $(X, \mathcal{T})$ be a topological space, and let A be a subset of X. Then $\mathcal{T}_A$ forms a topology on A. This is called the* ***subspace topology on*** *A.*

Proof. I leave it to you to verify the four axioms (T1)–(T4) for a topology. □

THEOREM 7.6. *Let $(X, \mathcal{T})$ be a topological space and let $(A, \mathcal{T}_A)$ be a subspace. Then the inclusion map $i : A \to X$ is continuous.*

Proof. If U is an open set in X then $i^{-1}(U) = U \cap A$ and $U \cap A \in \mathcal{T}_A$. □

EXAMPLE 7.7. Consider the Euclidean line $(\mathbb{R}, \mathcal{T}_{euclid})$, and the subsets

$$A_1 = [0, 4], \quad A_2 = [0, 1] \cup [3, 4] \quad \text{and} \quad A_3 = \mathbb{Z} \ .$$

We describe some aspects of the subspace topology on A_1, A_2 and A_3.
Subspace topology on A_1: Notice that the interval $[0, 1) \subset A_1$ is open in the subspace topology on A_1 since $[0, 1) = (-1, 1) \cap A_1$ and $(-1, 1)$ (being an open interval) is an open set in the Euclidean topology. On the other hand $[0, 1)$ is **not** an open set in the Euclidean topology on $\mathbb{R}$.
Subspace topology on A_2: Since $[0, 1] = (-1, 2) \cap A_2$ and $[3, 4] = (2, 5) \cap A_2$, the sets $[1, 2]$ and $[3, 4]$ are open sets in the subspace topology on A_2. Moreover, observe that $A_2 - [1, 2] = [3, 4]$ and it follows that $[0, 1]$ is a closed set in the subspace topology on A_2.
Subspace topology on A_3: Let n an element of $A_3 = \mathbb{Z}$. Then the open interval $(n - 1/2, n + 1/2)$ is an open set in the Euclidean line. Thus $(n - 1/2, n + 1/2) \cap \mathbb{Z} = \{n\}$ is an open set in the subspace topology on $\mathbb{Z}$. This shows that every singleton subset $\{n\}$ of $\mathbb{Z}$ is open in the subspace topology. If V is any subset of $\mathbb{Z}$ then $V = \cup_{n \in V}\{n\}$ so V is an open set by axiom (T3). This shows that the subspace topology on $\mathbb{Z}$ is the discrete topology.

When A is a subset of Euclidean n-space $\mathbb{R}^n$ then we refer to the subspace topology as the **Euclidean topology on** A.

THEOREM 7.8. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. Let A be a subset of X. Then the function $g : A \to f(A)$ defined by $g(a) = f(a)$ for each $a \in A$ is continuous when $A \subseteq X$ and $f(A) \subseteq Y$ have the subspace topologies.*

Proof. Let $f : X \to Y$ be a continuous, let $A \subseteq X$ and let $g : A \to f(A)$ be as defined. Suppose U is an open subset of $f(A)$ with the subspace topology. Then there is an open set U' in Y so that $U = U' \cap f(A)$. Then

$$\begin{aligned} g^{-1}(U) = \{a \in A \mid g(a) \in U\} &= \{a \in A \mid g(a) \in U'\} \\ &= \{x \in X \mid f(x) \in U'\} \cap A = f^{-1}(U') \cap A \ . \end{aligned}$$

Since $f^{-1}(U')$ is open in X by the continuity of f, it follows that $g^{-1}(U)$ is open in the subspace topology on A. Therefore $g : A \to f(A)$ is continuous. □

8. Connectedness and the Intermediate Value Theorem

DEFINITION 8.1. A topological space $(X, \mathcal{T})$ is said to be **connected** provided that the only subsets of X which are both open and closed are the empty set and the whole space X. (By axioms (T1) and (T2) we know that the sets $\emptyset$ and X are always both open and closed.)

Many authors give the definition of connectedness of X as: *if $X = U \cup V$ where U and V are open sets which are disjoint then either U or V is the empty set.* [Convince yourself of the equivalence of these definitions.]

EXAMPLE 8.2. The trivial topology $\mathcal{T}_{trivial}$ on a set X is connected since in this topology the only open sets are $\emptyset$ and X. If X is a set with at least two distinct elements then the discrete topology $\mathcal{T}_{discrete}$ on X is not connected: because if $x_0 \in X$ then the singleton set $\{x_0\}$ is both open and closed (in the discrete topology, every subset of X is both open and closed) but $\{x_0\} \neq \emptyset$ and $\{x_0\} \neq X$.

EXAMPLE 8.3. Consider the set A_2 from example 7.7. Since $[0, 1]$ is an open and closed set which is neither $\emptyset$ nor A_2, we conclude that A_2 with the subspace topology is not a connected topological space. This example should give you an idea of where the term 'connected' comes from. (Draw a picture of A_2 in $\mathbb{R}$, does it look like it should be 'connected'?) The discussion of this example $A_2 \subset \mathbb{R}$ leads directly to the next result.

THEOREM 8.4. *If A is a subset of $\mathbb{R}$ which is not an interval then A is not connected in the Euclidean topology.*

Proof. If A is not an interval then there is a real number $x_0 \in \mathbb{R} - A$ such that A contains at least one real number smaller than x_0 and at least one number larger than x_0. Then $(-\infty, x_0) \cap A$ is both open and closed in the subspace topology on A but it is neither the empty set nor all of A. $\square$

THEOREM 8.5. *Every interval in the real line is connected in the Euclidean topology.*

Proof. We will give the proof only in the case where the interval is a closed finite interval $[a, b]$ where $a < b$. It is not difficult to adapt the proof to handle the other possible intervals such as $\mathbb{R} = (-\infty, +\infty)$, $(-\infty, b]$, $[a, b)$ and so on.

Suppose that U is a nonempty subset of $[a, b]$ which is both open and closed in the Euclidean topology. In particular, this means that $[a, b] - U$ is open (since U is closed). By replacing U with $[a, b] - U$ if necessary, we may assume that $a \in U$. The goal of the proof is to show that $U = [a, b]$; if we can show this then it follows that $[a, b]$ is connected. Define a subset S of $[a, b]$ by

$$S = \{s \in [a, b] \mid [a, s] \subseteq U\}$$

Observe that $S \subseteq U$, that $a \in S$ (since $[a, a] = \{a\}$), and that if $s \in S$ and $a < t < s$ then $t \in S$. This shows that S must be an interval of the form $[a, m)$ or $[a, m]$ for some $m \in [a, b]$. We now consider three different cases.

CASE 1: $S = [a, m)$ and $m \notin U$. Since $m \notin U$, m is an element of $[a, b] - U$ which is an open

subset of $[a,b]$. It follows that there is an $\epsilon > 0$ such that $(m-\epsilon, m+\epsilon) \cap [a,b] \subseteq [a,b] - U$. If s is an element of $(m-\epsilon, m+\epsilon) \cap [a,b]$ which is smaller than m then $s \in S \subseteq U$, which is impossible since $m \notin U$. Therefore CASE 1 can't happen.
CASE 2: $S = [a,m)$ and $m \in U$. Then there is an $\epsilon > 0$ such that $(m-\epsilon, m+\epsilon) \cap [a,b] \subseteq U$. Let s be an element of $(m-\epsilon, m+\epsilon) \cap [a,b]$ which is smaller than m. Then $[a,s] \subseteq U$ (since $s \in S$) and $[s,m] \subseteq U$. This implies that $[a,m] = [a,s] \cup [s,m] \subseteq U$, and that $m \in S$, which is impossible. Therefore CASE 2 can't happen.
CASE 3: $S = [a,m]$ and $m < b$. In this case $m \in U$ since $m \in S$. So there is an $\epsilon > 0$ with $(m-\epsilon, m+\epsilon) \subseteq U$. Let s be an element of this open interval which is larger than m. Then $s \notin S$ (since $s > m$) but $s \in S$ (since $[a,s] = [a,m] \cup [m,s] \subseteq U$). This contradiction shows that CASE 3 can't happen.
Since none of the three cases can happen, we conclude that $S = [a,m]$ where $m = b$. By the definition of the set S this means that $U = [a,b]$, and the proof is complete. □

THEOREM 8.6. *Let X and Y be topological spaces and let $f : X \to Y$ be a function which is continuous and onto. If X is connected then Y is connected.*

Proof. Let U be a subset of Y which is both open and closed. Then $f^{-1}(U)$ is open in X since f is continuous. Also $f^{-1}(U)$ is closed in X by theorem 7.2. So $f^{-1}(U)$ is both open and closed. As X is connected we conclude that either (1) $f^{-1}(U) = \emptyset$ or (2) $f^{-1}(U) = X$. Using the hypothesis that f is onto, in case (1) U must be empty and in case (2), U must equal Y. Thus the only subsets U of Y that are both open and closed are $\emptyset$ and Y, so Y is connected. □

We can now prove the INTERMEDIATE VALUE THEOREM:

THEOREM 8.7. *If $f : [a,b] \to \mathbb{R}$ is continuous and c is a real number between $f(a)$ and $f(b)$ then there is an $x \in [a,b]$ such that $f(x) = c$.*

Proof. Let $f : [a,b] \to \mathbb{R}$ be continuous. Then the function $g : [a,b] \to f([a,b])$ defined by $g(x) = f(x)$ for $x \in [a,b]$ is continuous by theorem 7.8, and g is onto. We know that $[a,b]$ is connected with the Euclidean topology by theorem 8.5, and so $g([a,b]) = f([a,b])$ is connected by theorem 8.6. Therefore $f[a,b]$ must be an interval by theorem 8.4. Since $f(a)$ and $f(b)$ are elements of the interval $f[a,b]$, any real number c between $f(a)$ and $f(b)$ must be in $f([a,b])$. This completes the proof. □

This proof of the Intermediate Value Theorem readily extends to a more general statement, which has been applied in many different situations since Hausdorff's time. Most of these applications have to do with being able to guarantee that there will exist solutions to certain types of equations.

THEOREM 8.8. *Let X be a connected topological space and let $f : X \to \mathbb{R}$ be continuous with respect to the Euclidean topology on $\mathbb{R}$. If a and b are elements of X and c is a real number between $f(a)$ and $f(b)$ then there is an $x \in X$ such that $f(x) = c$.* □

9. Compactness and the Extreme Value Theorem

DEFINITION 9.1. Let $(X, \mathcal{T})$ be a topological space. An **open cover of** X is a collection $\mathcal{U}$ of open subsets of X such that $\bigcup \mathcal{U} = X$ where $\bigcup \mathcal{U} = \bigcup_{U \in \mathcal{U}} U$. In other words, $\mathcal{U}$ is an open cover of X iff $\mathcal{U} \subseteq \mathcal{T}$ and for each $x \in X$ there is at least one $U \in \mathcal{U}$ such that $x \in U$. If $\mathcal{U}$ is an open cover of X then a **subcover of** $\mathcal{U}$ is a subset $\mathcal{V} \subseteq \mathcal{U}$ which also covers X (that is $\bigcup \mathcal{V} = X$). An open cover is **finite** if it contains only finitely many elements.

DEFINITION 9.2. We say that the topological space $(X, \mathcal{T})$ is **compact** provided that every open cover of X has a finite subcover.

EXAMPLE 9.3. The cofinite topology $\mathcal{T}_{cofinite}$ on a nonempty set X is compact. Recall that

$$\mathcal{T}_{cofinite} = \{\emptyset\} \cup \{X - F \mid F \text{ is a finite subset of } X\} .$$

Suppose $\mathcal{U}$ is an open cover of X. Then one of the elements of $\mathcal{U}$ has the form $X - F$ where $F = \{x_1, \ldots, x_k\}$ is a finite subset of X. For each $i = 1, \ldots, k$, let U_i be an element of $\mathcal{U}$ with $x_i \in U_i$. Then $\{X - F, U_1, \ldots, U_k\}$ is a finite subcover of $\mathcal{U}$.

EXAMPLE 9.4. The Euclidean line $\mathbb{R}$ is not compact. For example, the collection of open intervals $\mathcal{U} = \{(n-1, n+1) \mid n \in \mathbb{Z}\}$ is an open cover of the Euclidean line which has no finite subcover. (This is because each integer n is contained in the interval $(n-1, n+1)$ but not in any other interval in $\mathcal{U}$. So any proper subset of $\mathcal{U}$ will not cover $\mathbb{R}$.)
For a different explanation, it can also be shown that the collection of open intervals $\{(-n, n) \mid n \in \mathbb{Z}^+\}$ forms an open cover of $\mathbb{R}$ with no finite subcover.

EXAMPLE 9.5. The half-open interval $[0, 3)$ is not compact with the Euclidean topology. To see this observe that $\mathcal{U} = \{[0, 3 - \frac{1}{n}) \mid n \in \mathbb{Z}^+\}$ is an open cover of $[0, 3)$ which has no finite subcover. (Note that $[0, 3 - \frac{1}{n})$ is open in the subspace topology on $[0, 3)$ since $[0, 3 - \frac{1}{n}) = (-1, 3 - \frac{1}{n}) \cap [0, 3)$. The union of any finite subset of $\mathcal{U}$ will not include $3 - \frac{1}{N}$ for some positive integer N.)

THEOREM 9.6. *Every closed finite interval* $[a, b]$ *in the real line is compact in the Euclidean topology.*

Proof. Let $\mathcal{U}$ be an open cover of $[a, b]$ with respect to the Euclidean topology on $[a, b]$. Let S be the subset of $[a, b]$ defined by

$$S \ = \ \{s \in [a, b] \mid \text{there is a finite subset } \mathcal{V} \subseteq \mathcal{U} \text{ with } [a, s] \subseteq \bigcup \mathcal{V}\} .$$

Then $a \in S$ and if $s \in S$ and $a < t < s$ then $t \in S$. It follows that S is an inerval of the form $[a, m]$ or $[a, m)$ for some $m \in [a, b]$ (in the second form $[a, m)$ we assume that $m > a$). We consider two cases:
CASE 1: $S = [a, m]$ where $m < b$. In this case m is an element of S so there is a finite subset $\mathcal{V} \subseteq \mathcal{U}$ with $[a, m] \subset \bigcup \mathcal{V}$. Let U be an element of $\mathcal{V}$ containing m. Then there is

an $\epsilon > 0$ such that $(m - \epsilon, m + \epsilon) \cap [a, b] \subseteq U$. If t is an element of $(m - \epsilon, m + \epsilon) \cap [a, b]$ larger than m then $[a, t] = [a, m] \cup [m, t] \subseteq \bigcup \mathcal{V}$. Therefore there is an element t in S which is larger than m but this can's happen since m was assumed to be the largest element of S. This contradiction shows that CASE 1 can't happen.
CASE 2: $S = [a, m)$ where $m > a$. Let U be an element of the open cover $\mathcal{U}$ that contains m. Then there is an $\epsilon > 0$ such that $(m-\epsilon, m+\epsilon) \cap [a, b] \subseteq U$. Choose $t \in (m-\epsilon, m+\epsilon) \cap [a, b]$ smaller than m. Then $t \in S$ so that there is a finite subset $\mathcal{V}$ in $\mathcal{U}$ with $[a, t] \subseteq \bigcup \mathcal{V}$. Then $[a, m] = [a, t] \cup [t, m]$ is contained in $\bigcup(\mathcal{V} \cup \{U\})$. Since $\mathcal{V} \cup \{U\}$ is a finite subset of $\mathcal{U}$, this implies that $m \in S$, which is a contradiction. Therefore CASE 2 can't happen.
As neither CASE 1 nor CASE 2 can happen, we conclude that $S = [a, m]$ where $m = b$. This means that $\mathcal{U}$ has a finite subcover, which shows that $[a, b]$ is compact. □

THEOREM 9.7. *If C is a subset of $\mathbb{R}$ which does not have a largest (or a smallest) element then C is not compact with the Euclidean topology.*

Proof. Let C be a subset of $\mathbb{R}$ which does not have a largest element. We consider two cases according to whether or not C is 'bounded above'. A subset of $\mathbb{R}$ is **bounded above** if it is a subset of $(-\infty, H)$ for some real number H.
Case 1: Suppose that C is not bounded above. Then $\{(-\infty, n) \cap C \mid n \in \mathbb{Z}^+\}$ is an open cover of C which can't have any finite subcover (since C is not bounded above).
Case 2: Suppose that C is bounded above. Then C has a least upper bound m. Note that $m \in C$ since C does not have a largest element. In this case $\{(-\infty, m - \frac{1}{n}) \cap C \mid n \in \mathbb{Z}^+\}$ is an open cover of C which has no finite subcover.
In either case, we conclude that C is not compact. A similar argument using lower bounds works if C does not have a smallest element. □

The next theorem categorizes exactly which subsets of $\mathbb{R}$ are compact with the Euclidean topology (take $n = 1$). A subset C of $\mathbb{R}^n$ is **bounded** if there exists a positive real number N so that $C \subset B(0, N)$. Note that $C \subset \mathbb{R}$ is bounded iff there are real numbers L and H such that $L \leq x \leq H$ for all $x \in C$. Here L is called a **lower bound for** C and H is an **upper bound for** C.

THEOREM 9.8 (Heine-Borel Theorem). *A subset C of Euclidean n-space $\mathbb{R}^n$ is compact in the Euclidean topology if and only if C is closed and bounded.* □

One part of the proof of the Heine-Borel Theorem follows from the next result.

THEOREM 9.9. *If X is a compact topological space and C is a closed subset of X then C is compact with the subspace topology.*

Proof. Assume that X is a compact topological space and that C is a closed subset of X. Let $\mathcal{U}$ be an open cover of C. Then for each $U \in \mathcal{U}$ there is a set U' which is open in X such that $U = U' \cap C$. Since $\mathcal{U}$ covers C, C is a subset of $\bigcup\{U' \mid U \in \mathcal{U}\}$. Since C is a closed subset of X, $X - C$ is open and therefore $\{U' \mid U \in \mathcal{U}\} \cup \{X - C\}$ is an open cover of X. Since X is compact this open cover has a finite subcover which can be taken to be $\{U' \mid U \in \mathcal{V}\} \cup \{X - C\}$ where $\mathcal{V}$ is a finite subset of $\mathcal{U}$. Since $X - C$ doesn't contain any elements of C, $\mathcal{V}$ must cover C. This shows that $\mathcal{V}$ is a finite subcover of $\mathcal{U}$ and that C is compact. □

When $n = 1$ it is not difficult to complete the proof of the Heine-Borel Theorem, however we will not pursue it here as we won't need the result in our subsequent discussions. It is also not difficult to prove Heine-Borel Theorem when $n > 1$ although this does rely on some knowledge of 'product topological spaces', we will describe this concept later.

THEOREM 9.10. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous function. If X is compact then $range(f) = f(X)$ is compact with the subspace topology coming from Y.*

Proof. Let $f : X \to Y$ be a continuous function where X is compact. Let $\mathcal{U}$ be an open cover of $f(X)$ using the subspace topology that $f(X)$ inherits as a subset of Y. For each $U \in \mathcal{U}$ there is an open set U' in Y such that $U = U' \cap f(X)$. Because f is continuous, each of the sets $f^{-1}(U')$ where $U \in \mathcal{U}$ is open in X. Moreover the collection $\{U' \mid U \in \mathcal{U}\}$ is an open cover of X since $f(X) \subset \bigcup_{U \in \mathcal{U}} U'$. Since X is compact this open cover of X has a finite subcover $\mathcal{V}$. Then the collection $\{U \in \mathcal{U} \mid f^{-1}(U') \in \mathcal{V})\}$ is a finite subcover of $\mathcal{U}$. This shows that $f(X)$ is compact. $\square$

We can now assemble a proof of the EXTREME VALUE THEOREM.

THEOREM 9.11. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function with respect to the Euclidean topologies on $[a, b]$ and $\mathbb{R}$. Then f has a maximum value and a minimum value.*

Proof. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with respect to the Euclidean topologies on $[a, b]$ and $\mathbb{R}$. By theorem 9.6 the closed finite interval $[a, b]$ is compact. By continuity of f (using theorem 9.10) this implies the range of f is compact with the Euclidean topology. By theorem 9.7 the range of f must have a largest and a smallest element, and by definition these are respectively maximum and minimum values for f on $[a, b]$. $\square$

This proof of the Extreme Value Theorem readily extends to a more general statement

THEOREM 9.12. *Let X be a compact topological space and let $f : X \to \mathbb{R}$ be a continuous function with respect to the Euclidean topology on $\mathbb{R}$. Then there are elements $L, H \in X$ such that $f(L) \leq f(x)$ and $f(H) \geq f(x)$ for all $x \in X$.* $\square$

10. Metric Spaces

DEFINITION 10.1. Let X be a set. A **metric on** X is a function $d : X \times X \to [0, +\infty)$ which satisfies the following properties:

(M1) $d(x, y) = 0$ if and only if $x = y$.
(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(M3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A metric d on X is sometimes referred to as a **distance function** on X, and $d(x, y)$ is called the distance from x to y. The third axiom (M3) is known as the **triangle inequality**. A fixed set X can have many different metrics on it. If d is metric on X then we say that (X, d) is a **metric space**. When it is clear which metric d is being considered then we sometimes say that X is a metric space.

EXAMPLE 10.2. The **Euclidean metric on the real line** $\mathbb{R}$ is the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by $d(x, y) = |x - y|$ for $x, y \in R$. [Verify that this forms a metric.] More generally, the **Euclidean metric on** $\mathbb{R}^n$ is the function d given by

$$d(x, y) = \left((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right)^{1/2}$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

DEFINITION 10.3. Let (X, d) be a metric space. Let $x \in X$ and let ϵ be a positive real number. The **open ball of radius ϵ centered at** x is the set

$$B(x, \epsilon) \;=\; \{y \in X \mid d(y, x) < \epsilon\}.$$

If there is any doubt as to which metric d is used to define an open ball, we write $B_d(x, \epsilon)$ in place of $B(x, \epsilon)$.

THEOREM 10.4. *Let (X, d) be a metric space. The collection of subsets of X given by*

$$\mathcal{T}_d = \{U \subseteq X \mid \text{ for each } x \in U \text{ there is } \epsilon > 0 \text{ such that } B(x, \epsilon) \subseteq U\}$$

forms a topology on X. This topology is called the ***metric topology on** X **associated with** d.*

Proof. We need to verify the four topology axioms (T1)-(T4). This is proof is exactly the same as the proof of theorem 3.2. ☐

For each $x \in X$ and each $\epsilon > 0$, the open ball $B(x, \epsilon)$ is an open set in the metric topology $\mathcal{T}_d$. To prove this suppose that $y \in B(x, \epsilon)$, and put $\delta = \epsilon - d(x, y)$ (which is a positive real number since $d(x, y) < \epsilon$). If $z \in B(y, \delta)$ then

$$d(z, x) \leq d(z, y) + d(y, x) < \delta + d(x, y) = \epsilon - d(x, y) + d(x, y) = \epsilon.$$

This shows that $B(y, \delta) \subseteq B(x, \epsilon)$ and completes the proof that $B(x, \epsilon)$ is an open set in the metric topology. On the other hand, it is not true that every open set in the metric topology is an open ball. So the open balls are very special examples of open sets. However, the definition of the metric topology in theorem 10.4 can be seen to be equivalent to saying that every open set in $\mathcal{T}_d$ is a union of open balls. [Can you verify this? Look back at theorem 5.3.] If $\mathcal{B}$ is a subset of a topology $\mathcal{T}$ with the property that every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$ then we say that $\mathcal{B}$ **is a basis for the topology** $\mathcal{T}$. So we can say that the collection of open balls

$$\mathcal{B} = \{B(x, \epsilon) \mid x \in X \text{ and } \epsilon > 0\}$$

forms a basis for the metric topology on X.

Here are some more examples of metric spaces.

EXAMPLE 10.5. Let X be any set and let $d : X \times X \to [0, \infty)$ be defined by taking $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. It is easy to verify that (X, d) is a metric space. Moreover notice that for $x \in X$, the open ball $B(x, \epsilon)$ is the singleton set $\{x\}$ if $\epsilon \leq 1$. This shows that every singleton subset of X is open in the metric topology, and so $\mathcal{T}_d = \mathcal{T}_{discrete}$. This metric on X is called the discrete metric.

Example 10.6. On $\mathbb{R}^2$, a metric ρ can be defined by the equation:

$$\rho(x, y) \ = \ \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. [Verify the metric space axioms.] With this metric, an open ball $B_\rho(x, \epsilon)$ is actually an open square with horizontal and vertical sides whose center is at x (the side length of the square is 2ϵ). Since every circle has an inscribed square and vice-versa, it can be shown that the metric topology $\mathcal{T}_\rho$ on $\mathbb{R}^2$ is the same as the Euclidean topology. The moral of this example is that different metrics on a set X may very well generate the same topology.

Here's a more careful explanation that $\mathcal{T}_\rho$ coincides with the Euclidean topology on $\mathbb{R}^2$. Let d be the standard Euclidean metric on $\mathbb{R}^2$ given by

$$d(x, y) \ = \ \left((x_1 - y_1)^2 + (x_2 - y_2)^2\right)^{1/2}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then it is not hard to verify that, for each $\epsilon > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we have

$$B_\rho(x, \epsilon) \supset B_d(x, \epsilon) \quad \text{and} \quad B_d(x, \epsilon) \supset B_\rho(x, \epsilon/\sqrt{2}) \ .$$

Now suppose that U is a subset of $\mathbb{R}^2$ which is open in the ρ–metric topology $\mathcal{T}_\rho$. By definition this means that for each $x \in U$ there is $\epsilon > 0$ such that $B_\rho(x, \epsilon) \subseteq U$. Then $B_d(x, \epsilon)$ is contained in U (since $B_d(x, \epsilon)$ is a subset of $B_\rho(x, \epsilon)$), and this shows that U is open in $\mathcal{T}_d = \mathcal{T}_{euclid}$. On the other hand, suppose that U is open in the Euclidean topology on $\mathbb{R}^2$. This means that for each $x \in U$ there is $\epsilon > 0$ so that $B_d(x, \epsilon) \subseteq U$. Then $B_\rho(x, \epsilon/\sqrt{2}) \subseteq U$ and so U is open in the ρ–metric topology $\mathcal{T}_\rho$. Thus $\mathcal{T}_\rho = \mathcal{T}_d = \mathcal{T}_{euclid}$, as claimed.

Example 10.7. If (X, d) is a metric space and A is a subset of X. Then the function $d' : A \times A \to [0, \infty)$ defined by $d'(x, y) = d(x, y)$ for $x, y \in A$ is a metric on A. Note that, for $x \in A$, $B_{d'}(x, \epsilon) = B_d(x, \epsilon) \cap A$. From this it follows that the metric topology $\mathcal{T}_{d'}$ on A is the same as the subspace topology coming from the metric topology $\mathcal{T}_d$ on X.

There are many advantages to being able to work with a metric space topology rather than an arbitrary topology on a space. One of these is that in metric spaces, many aspects of the topology can be described in terms of 'convergent sequences'. For example, if X is a metric space and A is a subset of X then an element $x \in X$ will be a limit point of A if and only if there is a sequence of elements in A which converges to x. Of course we haven't defined what a convergent sequence is but it can be defined just like in calculus. That is, a sequence (x_n) in the metric space (X, d) converges to $x \in X$ iff for each $\epsilon > 0$ there is a positive integer N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. In metric topologies, sequences can also be used to describe the notion of compactness—so in metric spaces certain basic topological definitions can be formulated in different ways.

11. The Metrizability Problem

The definition of 'metric space' and its associated topology was first introduced by Frechet about ten years before Hausdorff's more general abstract definition of 'topological space'. The early history of topology was dominated by a desire to understand the relationship between these two definitions. To explain more precisely, we say that a topological space $(X, \mathcal{T})$ is **metrizable** if there is a metric d on X so that $\mathcal{T} = \mathcal{T}_d$. So the basic questions became: *is every topological space metrizable? if not, are there some additional axioms on a topological space that guarantee metrizability?* In order to study these 'metrizability problems', many fundamental properties of topological spaces were discovered and interrelationships developed. For example these studies led to various 'separation axioms' and 'countability axioms' which play a role throughout the subject. In the 1920's, a Russian mathematician Pavel Urysohn used some of these ideas to solve a special version of the metrizability problem which we will describe below (without proof). Ultimately, in the early 1950's, a complete (although complicated) solution was found to the most general version of the metrizability problem, bringing an end to the early history of the subject.

Here are two of the 'separation axioms' which play a role in the metrizability problem. Let $(X, \mathcal{T})$ be a topological space. Then X is a **Hausdorff space** provided that whenever x_1 and x_2 are distinct elements of X then there are neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$. The topological space X is said to be **normal** provided that whenever C_1 and C_2 are disjoint closed sets in X then there are disjoint open sets U_1 and U_2 with $C_1 \subseteq U_1$ and $C_2 \subseteq U_2$. One can show that in a Hausdorff topological space every singleton set is closed, and this weaker property is commonly referred to as the **T_1–axiom**. Many authors take as part of the definition of a normal space that the space satisfy the T_1–axiom. In the presence of this axiom (but not otherwise) it is easy to see that every normal space is Hausdorff.

THEOREM 11.1. *Let (X, d) be a metric space. The metric topology on X is Hausdorff. In fact, it is normal and T_1.*

Proof. We will only prove the Hausdorffness. Suppose that (X, d) is a metric space and consider the associated metric topology on X. Let x_1 and x_2 be distinct elements of X. Set ϵ to be half the distance from x_1 to x_2. That is $\epsilon = d(x_1, x_2)/2$ which is a positive real number (by (M1) since $x_1 \neq x_2$). Then $x_1 \in B(x_1, \epsilon)$ and $B(x_1, \epsilon)$ is a neighborhood of x_1 (since open balls are open in the metric topology). Likewise, $B(x_2, \epsilon)$ is a neighborhood of x_2. If $z \in B(x_1, \epsilon) \cap B(x_2, \epsilon)$ then

$$d(x_1, x_2) \leq d(x_1, z) + d(z, x_2) < \epsilon + \epsilon = 2\epsilon = d(x_1, x_2),$$

and this shows that $d(x_1, x_2)$ is strictly less than $d(x_1, x_2)$ which is clearly impossible. From this contradiction we conclude that $B(x_1, \epsilon) \cap B(x_2, \epsilon)$ is empty, thus showing that the metric topology is Hausdorff. □

One consequence of this theorem is that a topological space which is not Hausdorff (or normal) can't possibly be metrizable. For example, the trivial topology on a set with more than one elements is not metrizable. Also, the cofinite topology on an infinite set is T_1 but not Hausdorff, so it is not metrizable either.

Urysohn proved the following remarkable theorem:

THEOREM 11.2 (Urysohn's Lemma). *Let X be a normal topological space, and let A and B be disjoint closed sets in X. Then there is a continuous function $f : X \to [0, 1]$ with $f(A) = 0$ and $f(B) = 1$ where the interval $[0, 1]$ has the Euclidean topology.* □

And he used this result to establish:

Theorem 11.3 (Urysohn's Metrization Theorem). *Let X be a normal T_1 topological space which has a basis $\mathcal{B}$ with countably many elements in it. Then X is metrizable.* $\square$

The condition that X has a basis with countably many elements is an example of one of the 'countability axioms' for a topological space. This particular condition is often called **second countability of** X. For the record, a set $\mathcal{B}$ is **countable** (or $\mathcal{B}$ has 'countably many elements') if and only if there is a surjective function from $\mathbb{Z}^+$ to $\mathcal{B}$. Another of the countability axioms (one that is weaker than second countability) is known as 'separability': a topological space is **separable** provided that there is a countable subset of X whose closure equals X. Urysohn's Metrization Theorem only solves the metrizability problem for separable topological spaces— the complete solution took another thirty odd years for mathematicians to complete.

[More precisely, what we are claiming here is that a separable metric space is second countable. In fact, a metric space is separable if and only if it is second countable. See if you can prove this.]

A key role in the proof of Urysohn's Metrization Theorem is played by the construction of 'product of topological spaces'. Since this definition is very important in any graduate level course in point set topology, we include it here for completeness. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. Then the collection of subsets of $X \times Y$ given by

$$\{Z \subseteq X \times Y \mid \text{ for each } (x,y) \in Z \text{ there are } U \in \mathcal{T}_1 \text{ and } V \in \mathcal{T}_2 \text{ with } (x,y) \in U \times V \subseteq Z)\ \}$$

forms a topology $\mathcal{T}_{product}$ called the **product topology** on $X \times Y$. Equivalently, the product topology is the topology for which the collection of sets

$$\{U \times V \mid U \in \mathcal{T}_1 \text{ and } V \in \mathcal{T}_2\}$$

forms a basis. The sets $U \times V$ where $U \in \mathcal{T}_1$ and $V \in \mathcal{T}_2$ are often called **open rectangles in** $X \times Y$. Therefore, each open rectangle in $X \times Y$ is an open set in the product topology, but not every open set need be an open rectangle. In general, a subset of $X \times Y$ is open in the product topology iff it can be expressed as a union (possibly infinite) of open rectangles.

12. The Homeomorphism Problem

As the basic principles of topology evolved another fundamental problem which is philosphically very different than the metrization problem emerged as an important central problem in the field. The problem is frequently referred to as 'the homeomorphism problem'. It has its roots in nineteenth century (pre-topology) mathematics and continues to have great relevance in today's mathematics. In order to describe the problem we first need to define what a 'homeomorphism' is.

Recall that a function $f : X \to Y$ is a **bijection** if and only if it is both one-to-one and onto. Associated with each bijection $f : X \to Y$ there is an **inverse function** $f^{-1} : Y \to X$ defined for each $y \in Y$ by taking $f^{-1}(y)$ to equal x where $f(x) = y$. (Note that such an element $x \in X$ exists since f is onto and it is unique since f is one-to-one.) The inverse function f^{-1}

is characterized by the facts that the composition $f^{-1} \circ f$ is the identity function 1_X on X and that the composition $f \circ f^{-1}$ is the identity function 1_Y on Y. In other words, for each $x \in X$, $f^{-1}(f(x)) = x$, and for each $y \in Y$, $f(f^{-1}(y)) = y$.

A **homeomorphism between topological spaces** X **and** Y is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous. Another way to define what a homeomorphism between X and Y is is to say that it is a bijection $f : X \to Y$ for which a subset U is open in X if and only if $f(U)$ is open in Y. Thus a homeomorphism from X to Y induces a bijection between the open sets in X and the open sets in Y, and this captures the idea that X and Y are essentially the same topological space with elements labelled differently. If there is a homeomorphism between X and Y then we say that X **and** Y **are homeomorphic spaces**. Being homeomorphic is easily verifed to be an equivalence relation on the collection of all topological spaces. This means that: (1) each topological space X is homeomorphic to itself (the identity function 1_X is a homeomorphism); (2) if X is homeomorphic to Y then Y is homeomorphic to X (if $f : X \to Y$ is a homeomorphism then $f^{-1} : Y \to X$ is a homeomorphism; and (3) if X is homeomorphic to Y and Y is homeomorphic to Z then X is homeomorphic to Z (if $f : X \to Y$ and $g : Y \to Z$ are homeomorphisms then so is $g \circ f : X \to Z$).

The Homeomorphism Problem can now be phrased as: *Given two topological spaces determine whether or not they are homeomorphic.*

A property $\mathcal{P}$ of a topological space X which is expressed entirely in terms of the topology of X is called a **topological property**. If X and Y are homeomorphic spaces and $\mathcal{P}$ is a topological property then X satisfies $\mathcal{P}$ if and only if Y does. So one way to show that two topological spaces are not homeomorphic is to find a topological property satisfied by one of the spaces but not the other. For example, the intervals $[-1, 1]$ and $(-1, 1)$ (with the Euclidean topology) cannot be homeomorphic because $[-1, 1]$ is compact but $(-1, 1)$ is not compact and being compact is a topological property. We have encountered a number of topological properties in our discssions. For example each of the following is a topological property: compactness, connectedness, Hausdorffness, metrizability.

Problem List:

PROBLEM 1. For each positive integer n (that is, $n \in \mathbb{Z}^+$) let

$$U_n = (-n, n)$$

and let

$$V_n = [n, n+1].$$

(So, for each n, U_n is an open interval in $\mathbb{R}$, and V_n is a closed interval in $\mathbb{R}$.) Determine each of the following:

1. $U_5 \cup V_6$
2. $U_5 \cap V_6$
3. $U_1 \cup U_3$
4. $U_1 \cap U_3$
5. $V_1 \cup V_2 \cup V_3 \cup V_4$
6. $V_1 \cap V_2 \cap V_3 \cap V_4$
7. $\bigcup \{V_n \mid n \in \mathbb{Z}^+\}$
8. $\bigcap_{n \in \mathbb{Z}^+} V_n$
9. $\bigcup \{U_n \mid n \in \mathbb{Z}^+\}$
10. $\bigcap \{U_n \mid n \in \mathbb{Z}^+\}$

PROBLEM 2. If A, B and C are the sets $A = \{1, 2\}$, $B = \{1, 3, 5\}$ and $C = \{4\}$ then list all of the elements of the Cartesian product $A \times B \times C$.

PROBLEM 3. Let A and B be sets. If $A \cup B$ is a subset of $A \cap B$ what does that imply about the relationship between the sets A and B? Clearly state your result as a **Theorem**, and then prove it using the definitions of intersection and union. (Suggestion: Use Venn diagrams as an aid to make your conjecture but leave them out of your final proof.)

PROBLEM 4. Find the domain, codomain and range of each of the following functions:

a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2 + \sin(x)$.

b) $g : [0, 4] \to \mathbb{R}$ given by $g(x) = 3 - 2x^2$.

c) $h : \mathbb{R} \to [-4, +\infty)$ given by $h(x) = \begin{cases} -3 & \text{if } x < -1 \\ -x^2 & \text{if } -1 \le x \le 1 \\ x + 3 & \text{if } x > 1 \end{cases}$

PROBLEM 5. Determine whether each of the three functions in the previous problem are one-to-one. If a particular function is not one-to-one then find two distinct elements of the domain that have the same image.

PROBLEM 6. Determine whether each of the following subsets of $\mathbb{R}^2$ is an open set, or a closed set. Give some justification for your two assertions (ie- whether it is open or not open, and whether it is closed or not closed) for each part.

a) $U = \mathbb{R}^2$.

b) $U = \{(x, y) \in \mathbb{R}^2 \mid y \leq 5\}$.

c) $U = \{(x, y) \in \mathbb{R}^2 \mid 2 < y \leq 5\}$.

d) $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 10\}$.

(Warning: It is not true that any set which is not open must be closed, so you have to independently check BOTH assertions in these problems.)

PROBLEM 7. Consider the set of real numbers $\mathbb{R}$. Let $\mathcal{T}$ be the collection of subsets of $\mathbb{R}$ which consists of $\emptyset$, $\mathbb{R}$ and every finite open interval (a, b) where $a < b$. In other terms,

$$\mathcal{T} \; = \; \{\emptyset, \mathbb{R}\} \cup \{(a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$$

a) Prove that $\mathcal{T}$ is not a topology on $\mathbb{R}$.

b) The collection $\mathcal{T}$ does satisfy three of the four axioms T1—T4. Determine which three axioms these are, and then explain why each of the three is true.

PROBLEM 8. Consider the function $f : \mathbb{R}^2 \to R$ given by $f(x_1, x_2) = x_1^2 + x_2^2$.

a) In the x_1x_2-plane draw the set $f^{-1}(B_1)$ where $B_1 = \{2, 3\} \subset \mathbb{R}$.

b) In the x_1x_2-plane draw the set $f^{-1}(B_2)$ where $B_2 \subset \mathbb{R}$ is the closed interval $[2, 3]$.

c) In the x_1x_2-plane draw the set $f^{-1}(B_3)$ where $B_3 \subset \mathbb{R}$ is the closed interval $[-3, -2]$.

d) Is there any real number t such that $f^{-1}(\{t\})$ is a singleton set? Explain. (Remember: A **singleton set** is a set with exactly one element in it. For example, $\{t\}$ is a singleton set.)

e) Is there any subset $B \subseteq \mathbb{R}$ for which $f^{-1}(B)$ is the open disk $B(x, \epsilon)$ where $x = (1, 2) \in \mathbb{R}^2$ and $\epsilon = 1$? Explain.

PROBLEM 9. Let $f : X \to Y$ be an onto function and let B be a subset of Y. Use the definition of image and inverse image to prove that $f(f^{-1}(B)) = B$. Did you need to use the hypothesis that f is onto?

PROBLEM 10. Consider the collection of subsets $\mathcal{T}_\ell$ of the real line $\mathbb{R}$ given by

$$\mathcal{T}_\ell \;=\; \{U \subseteq \mathbb{R} \mid \text{ for each } x \in U \text{ there is } \epsilon > 0 \text{ so that } [x, x+\epsilon) \subseteq U\}.$$

a) Show that $\mathcal{T}_\ell$ forms a topology on $\mathbb{R}$. (Suggestion: Use the proof of Theorem 3.3 in the class notes as a model.)

b) Show that every open set in the Euclidean topology $\mathcal{T}_{euclid}$ on $\mathbb{R}$ is also an open set in the $\mathcal{T}_\ell$ topology. In other words, $\mathcal{T}_{euclid} \subseteq \mathcal{T}_\ell$.

c) Find (and verify) an example of a set U that is open in the $\mathcal{T}_\ell$ topology but not open in the $\mathcal{T}_{euclid}$ topology.

The topology $\mathcal{T}_\ell$ described in this exercise is called the **lower limit topology on** $\mathbb{R}$. Parts (b) and (c) show that $\mathcal{T}_{euclid} \subsetneq \mathcal{T}_\ell$, so that the lower limit topology on $\mathbb{R}$ is strictly finer than the Euclidean topology on $\mathbb{R}$.

PROBLEM 11. Let X be a set and let x_0 be an element of X. Define $\mathcal{T}$ to be the collection of subsets of X consisting of X itself and all subsets of X which do not contain x_0. Thus

$$\mathcal{T} \;=\; \{X\} \cup \{U \subseteq X \mid x_0 \notin U\} \;.$$

a) Show that $\mathcal{T}$ forms a topology on X.

b) Describe all of the closed sets in this topology which contain x_0. Explain.

c) Show that if X has at least two elements then there is a subset of X which is closed in the $\mathcal{T}$ topology but not open.

The topology on X is called the **excluded point topology**.

PROBLEM 12. There's a topology $\mathcal{T}$ defined on the real line $\mathbb{R}$ by

$$\mathcal{T} \;=\; \{\emptyset, \mathbb{R}\} \cup \{(-a, a) \mid a \in \mathbb{R} \text{ and } a > 1\} \;.$$

(You can check that this is indeed a topology but that's not being asked as part of this problem.)

a) Describe the collection of closed sets in this topology on $\mathbb{R}$.

b) Determine the closure of each of the following subsets of $\mathbb{R}$ with respect to this topology: $A_1 = (2, 3]$, $A_2 = (0, 2)$, and $A_3 = \{\sqrt{2}\}$.

c) Show that the function $f : (\mathbb{R}, \mathcal{T}_{euclid}) \to (\mathbb{R}, \mathcal{T})$ given by $f(x) = x^2$ for $x \in \mathbb{R}$ is continuous.

d) Show that the function $g : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}_{euclid})$ given by $g(x) = x^2$ for $x \in \mathbb{R}$ is not continuous. (Hint: You just need to find one Euclidean open set U for which $g^{-1}(U)$ is not in $\mathcal{T}$.)

Problem 13. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces.

(a) Show that if $\mathcal{T}_1$ is the discrete topology then every function $f : X \to Y$ is continuous.

(b) Show that if $\mathcal{T}_2$ is the trivial topology then every function $f : X \to Y$ is continuous.

c) Show that any constant function from X to Y is continuous.

Problem 14. Show that the discrete topology $\mathcal{T}_{discrete}$ on a set X is compact if and only if X is a finite set.

Problem 15. Consider the lower-limit topology $\mathcal{T}_\ell$ on the real line $\mathbb{R}$.

(a) Show that the interval $A = [1, 3]$ (with the subspace topology from $\mathcal{T}_\ell$) is not compact by showing that $\{3\} \cup \{[1, 3 - 1/n) \mid n \in \mathbb{Z}^+\}$ is an open cover of A with no finite subcover.

(b) Show that the set $B = \{0\} \cup \{1/n \mid n \in \mathbb{Z}^+\}$ (with the subspace topology from $\mathcal{T}_\ell$) is compact.

Problem 16. Let X be a set. State and prove conditions on X under which each of the following topological spaces is connected:
(a) the discrete topology on X.
(b) the trivial topology on X.
(c) the cofinite topology on X.
(d) the lower limit topology on $X = \mathbb{R}$.