Math 4513

Problems involving Complex Numbers and Functions

9/29/04

Throughout these problems z denotes a complex number with rectangular form $z = x + yi$.

1. Give examples showing that sometimes $Arg(zw)$ equals $Arg(z) + Arg(w)$ and sometimes it doesn't.

2. What is $|z|$ and $arg(z)$ when $z = (\sqrt{3} - i)^6$.

3. Determine the polar form of $1 + i$. For each integer n express $(1 + i)^n$ in polar form. Then use your answer to describe the rectangular form for $(1 + i)^n$. Plot the complex numbers $(1 + i)^n$ for a few values of n and see if you can describe the general pattern.

4. If Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$ is squared we get
   $\cos 2\theta + i\sin 2\theta = e^{i\theta^2} = (\cos\theta + i\sin\theta)^2$.
   (a) Expand out the right hand side of this equation and explain how this proves the double angles formulas for cos and sin.
   (b) Look at $e^{i\theta^3}$ and determine triple angle formulas for sin and cos.

5. In the set of real numbers the equation $x^5 = 1$ has only one solution but in the set of complex numbers $z^5 = 1$ has more than one solution. Write out all of these complex solutions. (Suggestion: Start with z written in polar form.)

6. Let $f(z) = z^2$ and consider the associated mapping $w = z^2$ from the z-plane to the w- plane.
   (a) Determine where the positive (and negative) real axis in the z-plane gets mapped in the w-plane.
   (b) Repeat (a) with the positive and negative imaginary axis.
   (c) Where does the circle with radius r_0 centered at the origin in the z-plane get mapped?
   (d) Express the function $f(z) = f(x + iy)$ in rectangular coordinates (that is–if $w = z^2$ and we write $w = u + iv$ what do u and v equal in terms of x and y?), and use this to determine where the horizontal line $y = 3$ gets mapped.

7. Repeat the previous problem with the function $g(z) = 2iz$.