Answers to Midterm
Foundations of Analysis

PART I. Clearly indicate whether each of the following statements is TRUE or FALSE.

1. The function $f$ from $\mathbb{R}$ to $\{x \in \mathbb{R} \mid x \geq 0\}$ given by $f(x) = x^2$ is surjective.
   TRUE. For each $a \in \{x \in \mathbb{R} \mid x \geq 0\}$, $\sqrt{a}$ is a real number which satisfies $f(\sqrt{a}) = a$. (Note that the function $f$ is not injective since, for example, $f(-3) = f(3)$ but $3 \neq -3$.)

2. If $A$ is a countable set and $B \subseteq A$ then $B$ is a countable set.
   TRUE. See Theorem 1.3.9. Remember that a countable set is a set which is either finite or denumerable.

3. Let $S_1 = \{a, b, c, d\}$ and let $S_2 = \{-3, -1, 0, 3, 5\}$. Then $\{(c, -3), (d, 0), (a, -3), (b, 5)\}$ is a function from $S_1$ to $S_2$.
   TRUE. Each element of $S_1$ is the first coordinate of exactly one ordered pair in the set $\{(c, -3), (d, 0), (a, -3), (b, 5)\}$. (Observe that the function is not one-to-one since $c$ and $a$ are distinct elements in $S_1$ which get mapped to the same element of $S_2$.)

4. Let $S_1 = \{a, b, c, d\}$ and let $S_2 = \{-3, -1, 0, 3, 5\}$. The function $g : S_1 \to S_2$ defined by $g(a) = 0$, $g(b) = 3$, $g(c) = 5$, $g(d) = -1$ is a one-to-one function.
   TRUE. Distinct elements of $S_1$ get mapped to distinct elements of $S_2$. (Note that the range of the function $g$ is $\{-1, 0, 3, 5\} \subset S_2$ and $g$ is not surjective.)

5. Let $S_1 = \{a, b, c, d\}$ and let $S_2 = \{-3, -1, 0, 3, 5\}$. There exists a function from $S_2$ to $S_1$ which is one-to-one.
   FALSE. See Theorem B.1 of Appendix B (which we discussed in class). The idea is that since the domain $S_2$ has five elements and the codomain $S_1$ has four elements there will have to be at least two distinct elements in $S_2$ which get mapped to the same element in $S_1$.

6. If $f : A \to B$ and $g : B \to C$ are injective then $g \circ f : A \to C$ is injective.
   TRUE. Suppose $f$ and $g$ are injective. Let $x_1$ and $x_2$ be elements of $A$ with $g \circ f(x_1) = g \circ f(x_2)$. Then $g(f(x_1)) = g(f(x_2))$ and this implies that $f(x_1) = f(x_2)$ by the injectivity of $g$. Since $f$ is injective and $f(x_1) = f(x_2)$, we conclude that $x_1 = x_2$.

7. There exists a surjective function from $\mathbb{N}$ to $\mathbb{R}$.
   FALSE. $\mathbb{N}$ is countable (in fact, denumerable) while $\mathbb{R}$ is uncountable. See Theorem 1.3.10.

8. If $S$ is a set and $T$ is a subset of $S$ which is not equal to $S$ then there is no bijection from $T$ to $S$.
   FALSE. As an example consider: Let $S$ be the set of natural numbers and let $T$ be the set of even natural numbers. Then $T$ is a proper subset of $S$ but the function $f : T \to S$ given by $f(n) = n/2$ is a bijection.

PART II. Prove each of the following.

1. Use mathematical induction to show that $2^n > n$ for each natural number $n$.
   SOLUTION: Let $P(n)$ be the statement that “$2^n$ is larger than $n$”. We prove $P(n)$ by induction on $n$. Consider first the case where $n = 1$. Since $2^1 = 2$ is larger than 1, the statement $P(1)$ is true.
Now suppose that $\mathcal{P}(k)$ is true for some natural number $k$. This means that $2^k > k$. Then, since $k \geq 1$ we have $2^k > 1$ and,

$$2^{k+1} = 2 \cdot 2^k = 2^k + 2^k > k + 2^k > k + 1.$$  

(Here we've used an elementary fact about an ordered field $F$: If $a, b, c \in F$ and $a > b$ then $a + c > b + c$.) Therefore $2^{k+1} > k + 1$ which shows that the statement $\mathcal{P}(k+1)$ is true. It follows by mathematical induction that $\mathcal{P}(n)$ is true for each natural number $n$. QED

2. The "symmetric difference" of two sets $A$ and $B$ is defined by $A \Delta B = (A \cup B) \setminus (A \cap B)$.

Show that if $A \Delta B = \emptyset$ then $A = B$.

SOLUTION: Let $A$ and $B$ be sets with $A \Delta B = \emptyset$. Suppose $a$ is an element of $A$. Then $a \in A \cup B$ (since $A \cup B$ consists of all elements which are in $A$ or in $B$) but $a \notin A \Delta B$ (because $A \Delta B$ has no elements). Since

$$A \Delta B = (A \cup B) \setminus (A \cap B) = \{x \mid x \in A \cup B \text{ and } x \notin A \cap B\}$$

it follows that $a \in A \cap B$. In particular, $a \in B$ (since $A \cap B$ consists of those elements that are in both $A$ and $B$). Thus every element in $A$ is an element of $B$ which shows that $A \subseteq B$.

Suppose $b$ is an element of $B$. Then $b \in A \cup B$ but $a \notin A \Delta B$. It follows, as before, that $b \in A \cap B$. In particular, $b \in A$ which shows that $B \subseteq A$. Therefore $A = B$ since $A \subseteq B$ and $B \subseteq A$. QED

3. Let $F$ be a field. Show (by referring to the field axioms) that the multiplicative identity 1 in $F$ is unique. In other words, if $u \cdot a = a = a \cdot u$ for each $a \in F$ then $u = 1$.

SOLUTION: Let $F$ be a field and let $u$ be an element of $F$ for which $u \cdot a = a = a \cdot u$. Then taking $a = 1$ (the multiplicative identity of $F$ which exists by axiom (M3)) we have $u = u \cdot 1 = 1$. QED

4. Let $F$ be an ordered field. Show that if $a$ is positive and $a < b$ then $1/a > 1/b$.

SOLUTION: Let $F$ be an ordered fields with elements $a$ and $b$ where $a > 0$ and $b > a$. Then $b > 0$ by transitivity (Theorem 2.1.7(a)). So neither $a$ nor $b$ (which are both positive) can equal 0 by trichotomy (O3). This implies that the reciprocals $1/a$ and $1/b$ exist by axiom (M4).

Now $b > a$ implies that $b - a > 0$, and axiom (O2) implies that $ab > 0$ (since $a$ and $b$ are positive). Lemma 1 (below) then shows that $1/(ab) > 0$. By Lemma 2 (below) and axiom (O2) we have $1/a - 1/b = (b - a) \cdot 1/(ab) > 0$. This shows that $1/a > 1/b$. QED

**Lemma 1:** If $a > 0$ then $1/a > 0$.

**proof.** Suppose $a > 0$. By Theorem 2.1.8(a), $a(1/a) = 1 > 0$, and then Theorem 2.1.10 shows that $1/a > 0$ (since $a > 0$, conclusion (ii) can’t occur). QED

**Lemma 2:** $1/a - 1/b = (b - a) \cdot 1/(ab)$.

**proof.** $(ab)(1/a - 1/b) = (ab)(1/a) - (ab)(1/b) = (ba)(1/a) - a(b - 1/b) = b(a \cdot 1/a) - a - 1 = b - 1 - a = b - a$ from whence the lemma follows by Theorem 2.1.3(a). QED