1. Define the following in precise terms:
   (a) The triangle inequality.
   (b) The infimum of a set $A$ of real numbers.
   (c) The absolute value of a real number.
   (d) The Archimedean Principle.
   (e) The Squeeze (or Pinching) Lemma for sequences.

2. (a) Give an example of a set $A$ of real numbers which does not have a supremum (greatest lower
    bound).
   (b) Give an example of a set of rational numbers which is bounded but does not have a supremum
    in $\mathbb{Q}$.

3. Let $f : X \to Y$ be a function and let $A$ and $B$ be subsets of $X$ such that $f(A) = f(B)$.
   (a) Show that $A$ need not equal $B$.
   (b) If $f$ is one-to-one then $A$ does equal $B$.

4. (a) Define what it means for a sequence $\{a_n\}$ to be increasing and for it to be decreasing.
   (b) Show that there exists a sequence which is both increasing and decreasing.
   (c) If a sequence is increasing then it cannot diverge to $-\infty$.

5. Use mathematical induction to prove that $9^n - 1$ is divisible by 8 for every integer $n \geq 1$.

6. Let $F$ be a field.
   (a) Show that $a \cdot 0 = 0$ for each $a \in F$.
   (b) Show that $(-a)^{-1} = -a^{-1}$ for each $a \in F$.

7. Let $F$ be a field and $a, b \in F$.
   (a) Explain step-by-step using the field axioms that $a^2 - b^2$ equals $(a - b)(a + b)$.
   (b) If $F$ is an ordered field and $x$ and $y$ are elements with $x \geq 0$, $y \geq 0$ and $x^2 = y^2$ show that $x = y$.

8. Show that the sequence $\{2n^2 - 1\}$ converges. Use the definition of limit to verify your statement.

9. Let $A$ and $B$ be sets. State and prove an equivalent statement describing when $A - B$ is empty.

10. Let $a$ and $b$ be real numbers. Show that if $a$ is rational and $b$ is irrational then $a + b$ is irrational.

11. In this problem $f$ is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
    (a) Consider the closed intervals $I = [1, 2]$ and $J = [-1, 0]$. Describe the four sets $f(I)$, $f(J)$, $f^{-1}(I)$
        and $f^{-1}(J)$.
    (b) Prove or give a counterexample: If $C$ and $D$ are subsets of $\mathbb{R}$ with $f^{-1}(C) = f^{-1}(D)$ then $C = D$.

12. Let $f : X \to Y$ and $g : Y \to Z$ be functions.
    (a) If $g \circ f$ is 1–1 then $f$ is 1–1.
    (b) If $g \circ f$ is 1–1 then $g$ need not be 1–1.

13. Let $F$ be a field. Prove the following using the field axioms (A1)–(A7).
    (a) If $a$ and $b$ are elements of $F$ with $a + b = 0$ then $a = -b$. (HINT: $(a + b) + (-b) = a + (b + (-b))$.)
    (b) For each $a \in F$, $-(a) = a$.
    (c) For each $a \in F$, $-a = (-1) \cdot a$.

14. Let $F$ be an ordered field.
(a) Use mathematical induction to prove that \(a^n > 0\) for any positive integer \(n\) and element \(a \in F\) with \(a > 0\).
(b) If \(a \neq 0\) then \(a^2 > 0\).

15. (a) Show that the sequence \(\{\frac{(-1)^n}{n}\}\) converges to 0.
(b) Prove or give a counterexample: If \(\{a_n\}\) is any sequence then \(\{\frac{a_n}{n}\}\) converges to 0.

16. Prove DeMorgan’s Law that \(X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)\) for any three sets \(X\), \(A\) and \(B\).

17. Let \(F\) be a field. If \(a\) and \(b\) are elements of \(F\) that satisfy \(a \cdot b = a\) where \(a \neq 0\), then \(b = 1\). (hint: look at \(a^{-1} \cdot (a \cdot b)\)).

18. If \(U_1\) and \(U_2\) are open sets in \(\mathbb{R}\) then so is \(U_1 \cap U_2\).

19. Let \(F\) be an ordered field and let \(P \subset F\) be the set of positive elements.
   a) Referring only to the field axioms show that \(0 \cdot a = 0\) for each \(a \in F\).
   b) Let \(x\) be a nonzero element of \(F\). Referring only to the ordered field axioms show that \(x^2 > 0\).

20. Let \(A \subset \mathbb{R}\). A real number \(x \in \mathbb{R}\) is called a limit point for \(A\) if and only if every \(\epsilon\)-neighborhood of \(x\) contains a point of \(A\) other than \(x\) itself. Show that \(x\) is a limit point for \(A\) if and only if every neighborhood of \(x\) contains infinitely many elements of \(A\).

21. Let \(f : X \to Y\) be a function. Let \(A \subset X\) and \(B \subset Y\).
   a) Show that \(f(f^{-1}(B)) = B\).
   b) Show that \(A \subset f^{-1}(f(A))\).
   c) Give an example where \(A \neq f^{-1}(f(A))\).

22. Let \(a, b \in \mathbb{R}\). Show that \(||a| - |b|| \leq |a - b|\). (Hint: one approach would be to consider four cases.)

23. Give examples of:
   a) A function \(f : \mathbb{R} \to \mathbb{R}\) which is both 1–1 and onto.
   b) A function \(f : \mathbb{R} \to \mathbb{R}\) which is neither 1–1 nor onto.
   c) A function \(f : \mathbb{R} \to \mathbb{R}\) which is 1–1 but not onto.
   d) A function \(f : \mathbb{R} \to \mathbb{R}\) which is onto but not 1–1.
   e) A sequence which has exactly three subsequential limit points.

24. Use the definition of open set to show that the interval \((3, 5)\) is an open set but that the interval \([3, 5]\) is not an open set.

**PART II.** Clearly indicate whether each of the following statements is TRUE or FALSE.

1. The union of two intervals is an interval.
2. If a sequence diverges then every subsequence of the sequence diverges.
3. \(x\) is a limit point of a set \(A \subset \mathbb{R}\) iff there exists \(\epsilon > 0\) so that \((x - \epsilon, x + \epsilon)\) contains infinitely many elements of \(A\).
4. If $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = \cos(x)$ then the set $f^{-1}([0,1])$ is bounded.
5. A sequence which is not bounded below cannot diverge to $\infty$.
6. A nonempty countable set cannot be open.
7. A set is open if and only if its complement is closed.
8. Every open set is an interval.
9. Any sequence has at most countably many subsequential limits.
10. Let $A$ be an infinite set which is bounded above. Then the least upper bound of $A$ is a limit point of $A$.
11. The set $\{1/n \mid n \in \mathbb{N}\}$ is closed.
12. The sequence $a_n = \sin(n^3)/n$ is convergent.
13. Every open set has a supremum.
14. Every finite subset of $\mathbb{R}$ is closed.
15. The set $\{z \in \mathbb{R} \mid |z - 3.5| < 2.5\}$ coincides with the open interval $(1,6)$.
16. Let $x$ and $y$ be real numbers and $\epsilon > 0$. The $\epsilon$-neighborhoods of $x$ and $y$ have nonempty intersection if and only if $|x - y| \leq \epsilon$.
17. Every subsequence of a constant sequence is constant.
18. The sequence $(x_k)$ where $x_k$ equals 1 if the natural number $k$ is a perfect square and equals $-1$ if $k$ is not a perfect square is a subsequence of the sequence $((-1)^n)$.
19. Every sequence in the interval $I = [1, \infty)$ has a subsequence which converges to an element of $I$.
20. Every sequence in the interval $I = (-1, 2]$ has a subsequence which converges to an element of $I$. 