Notes from 12/4 — Part 2

Surface Parametrization

Let S be a surface in 3-space described by:

$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, (u,v) in D. ← vector form

The importance of $\vec{r}_u \times \vec{r}_v$ in describing S:

fact 1 $\vec{r}_u \times \vec{r}_v (u_0,v_0)$ is perpendicular to S at $\vec{r}(u_0,v_0)$.

(because $\vec{r}_u(u_0,v_0)$ and $\vec{r}_v(u_0,v_0)$ are tangent to S.)

fact 2 $\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$

example $S: \begin{cases} x = u \\ y = v \\ z = u^2+v^2 \end{cases}$ where (u,v) is in the disk of radius 2 centered at origin in uv-plane

$\vec{r}(u,v) = \langle u, v, u^2+v^2 \rangle$

$\vec{r}_u(u,v) = \langle 1, 0, 2u \rangle$

$\vec{r}_v(u,v) = \langle 0, 1, 2v \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = -2u\vec{i} - 2v\vec{j} + \vec{k} \quad **$$

$\vec{r}(1,0) = \langle 1,0,1 \rangle$. So $(1,0,1)$ is a point on the surface S.

① Find eqt. for tangent plane to S at $(1,0,1)$

$\vec{r}_u \times \vec{r}_v (1,0)$ = normal vector to this plane

$$\parallel$$

$-2\vec{i} + \vec{k} = \langle -2, 0, 1 \rangle = \vec{N}$

Tangent plane goes thru $(1,0,1)$ and $\vec{N} = \langle -2,0,1 \rangle$

$-2(x-1) + 0(y-0) + 1(z-1) = 0$

$-2x + z + 1 = 0$

$2x - z = 1$ ← This is a plane that contains the y-axis.

② What is the surface area of S?

$\text{Area}(S) = \iint_D |-2u\vec{i} - 2v\vec{j} + \vec{k}| \, dA$

$= \iint_D \sqrt{(-2u)^2 + (-2v)^2 + 1^2} \, dA$

$= \iint_D \sqrt{4u^2 + 4v^2 + 1} \, dA$

$$\parallel$$

$\int_0^{2\pi} \int_0^2 \sqrt{4r^2+1} \, r \, dr \, d\theta$

$$\parallel$$

$\vdots$

express D in polar form

$\begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases}$

$u^2 + v^2 = r^2$

$dA = r\,dr\,d\theta$

$\begin{cases} u = 4r^2+1 \\ du = 8r\,dr \end{cases}$

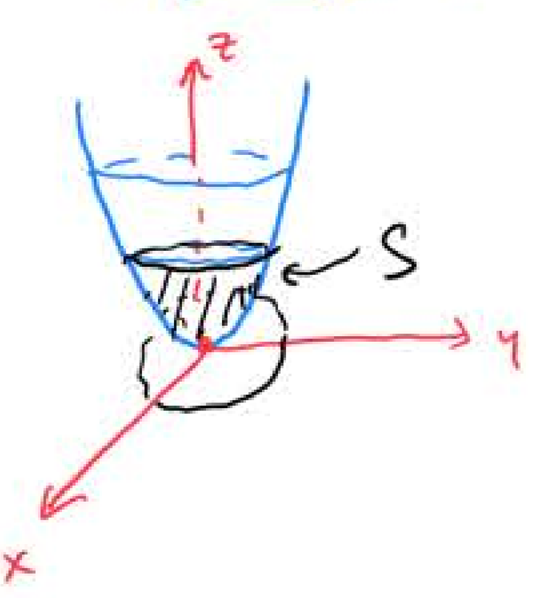
③ What does the surface look like?

$S: \begin{cases} x = u \\ y = v \\ z = u^2+v^2 \end{cases}$ (u,v) in D.

$u = x,\ v = y \Rightarrow z = u^2 + v^2 = x^2 + y^2$

eliminate parameters u and v: $z = x^2 + y^2$

S is part of the graph of $f(x,y) = x^2 + y^2$ in xyz-space.



circular paraboloid

where (u,v) in D

$$\parallel$$

(x,y)

This suggests:

General Example

$S: \begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$ ← Parametrization for the surface $z = f(x,y)$

surface satisfies VLP

The surface integral of $f(x,y,z)$ over a parametrized surface S

$S: \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

(u,v) in D

Define

$f(x(u,v), y(u,v), z(u,v))$

$$\iint_S f(x,y,z) \, dS = \iint_D f(x,y,z) \, |\vec{r}_u \times \vec{r}_v| \, dA$$

surface integral with respect to surface area

rubric $dS = |\vec{r}_u \times \vec{r}_v| \, dA$

similar to $ds = |\vec{r}\,'(t)| \, dt$