(1) Restriction Functions:

If $f : X \to Y$ is a function and $A \subseteq X$ then the "restriction of f to A" is the function

$$f|_A : A \to Y$$

domain$(f|_A) = A$
domain$(f) = X$

defined by $f|_A(x) = f(x)$ for $x \in A$.

Also, if $B \subseteq Y$ satisfies that $f(A) \subseteq B$ then we can view the restriction as

means

$$f|_A : A \to B.$$

$a \in A \Rightarrow f(a) \in B$

(This restriction function will be onto if $f(A) = B$.)

range(f)

Claim If f is injective then so is $f|_A$.

quick If $f|_A(a_1) = f|_A(a_2)$ then $f(a_1) = f(a_2)$. But f is injective and so $a_1 = a_2$.

(2) Algebra of functions.

For a set X consider the set $\mathcal{F}(X)$ consisting of all functions from X to itself.

$$\mathcal{F}(X) = \{ f : X \to X \}.$$

Observe $|\mathcal{F}(X)| = n^n$ if $|X| = n$.

Composition of functions defines an operation on $\mathcal{F}(X)$.

$$f, g \in \mathcal{F}(X) \Rightarrow f \circ g \in \mathcal{F}(X)$$

(where $f \circ g(x) = f(g(x))$).

One element of $\mathcal{F}(X)$ is the identity function

$$id_X : X \to X$$

defined by $id_X(x) = x$ for all $x \in X$.

Observe For any $f \in \mathcal{F}(X)$, $id_X \circ f = f = f \circ id_X$.

If $f \in \mathcal{F}(X)$ is a bijection then $f^{-1} \in \mathcal{F}(X)$ and $f \circ f^{-1} = id_X = f^{-1} \circ f$.

An interesting subset of $\mathcal{F}(X)$ is

$$\text{Perm}(X) = \{ f \in \mathcal{F}(X) \mid f \text{ is a bijection} \}$$

Observe that $|\text{Perm}(X)| = n!$ if $|X| = n$.

Some properties:

- $id_X \in \text{Perm}(X)$
- $f \in \text{Perm}(X) \Rightarrow f^{-1} \in \text{Perm}(X)$
- $f, g \in \text{Perm}(X) \Rightarrow f \circ g \in \text{Perm}(X)$
- $f, g, h \in \text{Perm}(X) \Rightarrow (f \circ g) \circ h = f \circ (g \circ h)$

In fact, with this operation of composition, the set Perm(X) forms an algebraic system known as a 'group'. Groups are one of main objects of study in the broad field of mathematics known as 'algebra'. You can learn more about this in the courses 'Abstract Algebra I and II'.