Consider the set $\mathbb{Z}_{30}$ of integers mod 30.

In $\mathbb{Z}_{30}$ the equations:

$7 \times 13 = 1$ (because $7 \cdot 13 = 3 \cdot 30 + 1$)

$11 \times 11 = 1$ (because $11 \cdot 11 = 4 \cdot 30 + 1$)

show that 13 is the multiplicative inverse of 7, 7 is the multiplicative inverse of 13, and 11 is its own multiplication inverse. So in $\mathbb{Z}_{30}$ it is legimate to write

$$\frac{1}{7} = 13, \quad \frac{1}{13} = 7 \quad \text{and} \quad \frac{1}{11} = 11.$$

On the other hand, $\frac{1}{6}$ = DNE because $6 \cdot 10 = 0$ in $\mathbb{Z}_{30}$.

(Explanation: If $\frac{1}{6}$ did exist in $\mathbb{Z}_{30}$ then

$$10 = \left(\tfrac{1}{6} \times 6\right) \cdot 10 = \tfrac{1}{6}(6 \times 10) = \tfrac{1}{6} \times 0 = 0,$$

which is a contradiction because 0 and 10 are two distinct elements of $\mathbb{Z}_{30}$.)

By similar reasoning, $\frac{1}{10}$ = DNE also.

So, which elements of $\mathbb{Z}_{30}$ do have multiplicative inverses?

<u>Answer</u>: The elements of $\mathbb{Z}_{30}$ with multiplicative inverses are the integers n between 0 and 29 for which the greatest common divisor of n with 30 is equal to 1. These are

$$1, 7, 11, 13, 17, 19, 23, \text{ and } 29.$$

(note: If n has a multiplicative inverse then so does $-n$. In fact $\frac{1}{-n} = -\frac{1}{n}$. In $\mathbb{Z}_{30}$ we have $29 = -1$, $23 = -7$, $19 = -11$, $17 = -13$.

So, for example, $\frac{1}{23} = -\frac{1}{7} = -13 = 17$ and

$$\frac{1}{19} = -\frac{1}{11} = -11 = 19.\ )$$