Determine injectivity / surjectivity for the following:

① $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $f(m,n) = m-n+4$.

one-to-one? No $\quad f(m_1,n_1) = m_1 - n_1 + 4$, $\quad f(m_2,n_2) = m_2 - n_2 + 4 \Rightarrow$

$$m_1 - n_1 + 4 = m_2 - n_2 + 4$$

$$m_1 - n_1 = m_2 - n_2 \qquad 4-1 = 10-7$$

$$\boxed{f(4,1) = 4-1+4 = 7, \quad f(10,7) = 10-7+4 = 7 \Rightarrow \text{ but } (4,1) \neq (10,7)}$$

onto? No Does every integer k equal $m-n+4$ for some (m,n)? $\quad m-n+4 = k$, $m-n = k+4$, $n=0$, $m = k+4$

final work $\quad \boxed{f(k+4,0) = (k+4)-(0)+4 = k}$

scratch work

② $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, $f(m,n) = 3m-6n+1$ $\quad$ one-to-one No

onto? No $\quad 3m-6n+1 \leftarrow$ Has a remainder of 1 when divided by 3.

$\{\ldots -5, -2, 1, 4, 7, 10, 13 \ldots\} = \text{range}(f) \neq \mathbb{Z}$

$f(m,n)$ never equals 0: If $3m-6n+1 = 0$ then $-1 = 3m-6n = 3(m-2n)$ but -1 is not divisible by 3.

③ $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, $f(m,n) = (3m, 2n+1)$

one-to-one? Yes $\quad f(m_1,n_1) = f(m_2,n_2)$

$$(3m_1, 2n_1+1) \qquad (3m_2, 2n_2+2)$$

$$\begin{cases} 3m_1 = 3m_2 \\ 2n_1+1 = 2n_2+1 \end{cases} \Rightarrow \begin{cases} m_1 = m_2 \\ n_1 = n_2 \end{cases} \Rightarrow (m_1,n_1) = (m_2,n_2)$$

Be sure that you have indicated your logical process!!

onto? No $\quad$ 2nd coordinate of $(3m, 2n+1)$ is always odd.

④ $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$, $f(m,n) = (m-n, m+1)$

one-to-one? Yes $\quad f(m_1,n_1) = f(m_2,n_2)$

$$(m_1-n_1, m_1+1) \qquad (m_2-n_2, m_2+1)$$

$$\begin{cases} a \quad m_1 - n_1 = m_2 - n_2 \\ b \quad m_1 + 1 = m_2 + 1 \end{cases} \qquad (b) \Rightarrow m_1 = m_2$$

And then (a) gives $m_1 - n_1 = m_1 - n_2$ $\Rightarrow n_1 = n_2$. Conclude: $(m_1,n_1) = (m_2,n_2)$

onto? Yes $\quad$ Can every $(k,\ell) \in \mathbb{Z}\times\mathbb{Z}$ equal $(m-n, m+1)$ for some (m,n)?

scratch

$\ell = m+1 \rightsquigarrow$ choose $m = \ell - 1$

$m-n = k \rightsquigarrow$ choose $n = m-k = \ell-k-1$

$$\boxed{f(\ell-1, \ell-k-1) = ((\ell-1)-(\ell-k-1), (\ell-1)+1) = (k,\ell)}$$