Equivalence Relations

A relation $\equiv$ on a set S is an equivalence relation provided that it is

(i) reflexive: If $s \in S$ then $s \equiv s$.

(ii) symmetric: If $s, t \in S$ and $s \equiv t$ then $t \equiv s$.

(iii) transitive: If s, t and u are elements of S where $s \equiv t$ and $t \equiv u$ then $s \equiv u$.

Examples ① On any set S, $=$ is an equivalence relation.

② For each $m \in \mathbb{N}$, $\equiv_m$ (congruence mod m) is an equivalence relation on the set $\mathbb{Z}$ of integers.

③ On the set $\hat{\mathbb{Q}}$ of formal quotients with form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$, write $\frac{p}{q} \equiv \frac{p'}{q'}$ provided that $\frac{p}{q}$ and $\frac{p'}{q'}$ are equal as real numbers. Another way to write this is: $\frac{p}{q} \equiv \frac{p'}{q'}$ means $pq' = qp'$.

④ On the set of all directed line segments in the plane, define $\overrightarrow{PQ} \equiv \overrightarrow{RS}$ if the directed line segments $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are parallel, have the same length, and point in the same direction.

If $\equiv$ is an equivalence relation on a set S then we can define a new set $S/\equiv$. Each element of $S/\equiv$ is represented by an element of S with the understanding that $s, t \in S$ represent the same element of $S/\equiv$ whenever $s \equiv t$.

Examples from above continued:

① $S/=$ is the set S itself.

② $\mathbb{Z}/\equiv_m$ is the set $\mathbb{Z}_m$ of integers mod m.

③ $\hat{\mathbb{Q}}/\equiv$ is the set of rational numbers.

④ $\{\overrightarrow{PQ} \mid P \text{ and } Q \text{ are points in the plane}\}/\equiv$ is the set of vectors in the plane.
