1. How many 6 letter combinations with repetition can made using the letters $\{a, b, c, d\} = X$?

So $|X| = 4 = n$ and $k = 6$

Some examples:

aabdca ← equal as combinations

" badaca

" aaabcd

$\#a = 3$ $\#c = 1$

$\#b = 1$ $\#d = 1$

To solve the problem we must indicate the # of a's, # of b's, # of c's, # of d's.

Consider strings of $*$'s and $|$'s (stars and bars) with 3 bars and 6 stars.

compartment 2 ← compartment 4

$**|*|*|*$ ← string of length 9 with 3 bars and 6 stars.

compartment 1

compartment 3

a b c d

$***|*|*|*$ ⤳ encodes that we have 3 a's, 1b, 1c, 1d.

$|***|*|**$ ← 0 a's, 3 b's, 1c, 1d

a b c d

How many strings of length 9 with 3 bars and 6 stars are there?

aacdd

a b c d

$* \; * \; | \; | \; * \; * \; | \; * \; *$

Out of 9 possible slots choose a set of 3 slots in which to put bars.

There are $\binom{9}{3} = \frac{9!}{6!3!} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

2. How many k letter combinations with repetition can be made choosing letters from an n-element set?

Consider strings with $n-1$ bars and with k stars. These strings have length $n-1+k$.

From $n-1+k$ slots choose $n-1$ positions to put the bars. The number of ways to do this is $\binom{n+k-1}{n-1}$.

3. How many ways can 20 be written as a sum of 5 non-negative integers $20 = n_1 + n_2 + n_3 + n_4 + n_5$?

$$n_1 + n_2 + n_3 + n_4 + n_5 = 20 \quad (*)$$

Consider strings with 4 bars and 20 stars

examples $6 + 1 + 0 + 1 + 12 = 20$

$******|*||*|************$

6 1 0 1 12

Answer There are $\binom{24}{4} = \frac{24!}{4!\,20!} = \frac{24 \cdot 23 \cdot 22 \cdot 21}{4 \cdot 3 \cdot 2 \cdot 1} = 10{,}626$

$1 + 0 + 6 + 12 + 1 = 20$

$*||******|************|*$

4. How many ways can we write $20 = n_1 + n_2 + n_3 + n_4 + n_5$ if each $n_k \in \mathbb{N}$? (Not allowing 0's)

$$n_1 + n_2 + n_3 + n_4 + n_5 = 20$$

$$(n_1 - 1) + (n_2 - 1) + (n_3 - 1) + (n_4 - 1) + (n_5 - 1) = 15$$

Observe that each $n_k - 1 \geq 0$.

Answer Consider strings with 15 $*$'s and 4 bars.

There are $\binom{19}{4} = \frac{19!}{4!\,15!} = 3{,}876$