On the square root of two
Discrete Math, Fall 2020

Consider a right triangle whose two non-hypotenuse sides have length one. By the Pythagorean Theorem the square of the length of the hypotenuse of this triangle equals two. Since real numbers consist of all distances between points in a plane, this observation confirms that there is a positive real number whose square is two. This real number is denoted by $\sqrt{2}$.

A **rational number** is a number which can be expressed as the quotient of two integers in which the denominator is not equal to 0. An **irrational number** is a real number which is not rational.

THEOREM 1. *The square root of* 2 *is irrational.*

Proof. Suppose that the statement of the theorem is false. Then there must be a rational number x with $x^2 = 2$. As every rational number has a unique reduced fraction representative, we can write $x = p/q$ where p and q are integers with no common divisors other than -1 and 1. Since $x = p/q$ we know that $p = xq$ and squaring both sides of this equation gives that

$$p^2 = (xq)^2 = x^2q^2 = 2q^2 \tag{1}$$

(remembering that $x^2 = 2$). As q^2 is an integer, it follows that p^2 is an even integer, and this implies that p itself must be even. (See the lemma below.) Since p is even we can write $p = 2n$ for some integer n. Substituting this into equation (1) shows that

$$2q^2 = p^2 = (2n)^2 = 4n^2.$$

Dividing both sides of this equation by 2, we see that $q^2 = 2n^2$. Therefore q^2 is an even integer, and, as before, we conclude that q must be an even integer. (See the lemma below.) This shows that both p and q are even, and it follows that 2 is a common divisor of both p and q. However this is impossible because we assumed that p and q have no common factors other that -1 and 1. In this argument, the only supposition that was made occurred in the first sentence where it was assumed that the statement of theorem is false. We conclude that the statement of theorem must be true. □

In our proof of the theorem we used some basic principles regarding even and odd integers. Here are some details of this.

An integer n is said to be **even** provided that there is an integer k for which $n = 2k$, and n is said to be **odd** provided that there is an integer ℓ for which $n = 2\ell + 1$.

One well-known fact about even and odd integers is that every integer is either even or odd, and no integer is both even and odd. This is not difficult to verify but we will take it as a known result here.[1]

LEMMA 1. *If the square of an integer is even then the integer itself is even.*

Proof. Suppose that the statement of the lemma is false. This means that there is an integer n for which n^2 is even but n is not even. Therefore n is odd and we can write $n = 2\ell + 1$ for some integer ℓ. Using standard rules of algebra we see that

$$n^2 = (2\ell + 1)^2 = (2\ell + 1)(2\ell + 1) = 4\ell^2 + 4\ell + 1 = 2(2\ell^2 + 2\ell) + 1 = 2L + 1$$

[1]This result may be viewed as a special case of a more general result, called the "division algorithm", that we will encounter later in the semester. The division algorithm says: *If n and m are integers and $m > 0$ then there are unique integers q and r such that $n = qm + r$ and $0 \le r < m$.* (The number r is called the "remainder" when n is divided by m, and q is called the "quotient".) Here the case of interest is when $m = 2$.

where $L = 2\ell^2 + 2\ell$. Observe that L is an integer since it is obtained from integers by performing multiplications and addition. Since $n^2 = 2L + 1$ and L is an integer we see that n^2 must be odd (using the definition of 'odd integer'). This shows that the integer n^2 is both even and odd, which is impossible. Therefore the supposition that the statement of the lemma is false must be incorrect, and we conclude that the lemma is true. $\square$