Math 2513
Some review for Exam 1 — with some answers at the bottom

Problem 1. *Give an elementwise proof that* $(A - B) \cap (A - C)$ *is a subset of* $\overline{B}$ *for all sets* A*,* B *and* C*.*

Problem 2. *Consider the proposition 'The square of an even number is an even number'.*
(a) Describe this proposition as an implication statement.
(b) Give the contrapositive statement..
(c) Give the converse statement.
(d) Which of (a),(b), (c) are true and which are false.

Problem 3. *Use a proof by contradiction to show that if* A*,* B *and* C *are sets and* A *is a subset of* C *then*

$$A \cap \overline{(B \cup C)} = \emptyset\ .$$

Problem 4. *Is it true for all sets* A*,* B *and* C *that* $\overline{(\overline{C} \cup (A \cap B))}$ *is a subset of* $A \cap C$*? Either prove your assertion or give a counterexample.*

Problem 5. *Use an elementwise approach to prove the statement: For all sets* A*,* B *and* C*,* $(B \cup C) - A = (B - A) \cup (C - A)$*.*

Problem 6. *Determine if each statement is true or false. Provide justification as appropriate.*
(a) $\{a\} \in \{\{a\}\}$
(b) $\mathbb{Z} \in \mathbb{Z}$
(c) $\mathbb{Z} \subseteq \mathbb{Z}$.
(d) $\varnothing \subseteq \{\varnothing\}$.
(e) $\{\varnothing\} \subseteq \varnothing$.
(f) $\{\{\{\varnothing\}\}\} \subseteq \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$.
(g) The set $\{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$ *has cardinality 4.*
(h) If P *and* Q *are statements then* $\neg(P \implies Q)$ *is logically equivalent to* $\neg P \implies \neg Q$*.*

Problem 7. *Let* A *and* B *be sets. State the negation of each of the following propositions as directly as possible.*
(a) B *is a subset of* A *or* $A \cap B = \emptyset$*.*
(b) If B *is a subset of* A *then* A *and* B *are disjoint.*
(c) For every x *in* A*,* B *contains an element other than* x*.*

Problem 8. *Let* A *be set whose power set* $\mathcal{P}(A)$ *has* 32 *elements.*
(a) What is the cardinality of A*?*
(b) Does $\mathcal{P}(\mathcal{P}(A))$ *have more than a million elements? Explain.*

Problem 9. *For sets* A*,* B *and* C *let*

$$X = A \cap \big((B \cup C) - (B \cap C)\big) \cap B \quad \text{and} \quad Y = (B - C) \cap A.$$

(a) If x *is an element of* $A \cap B \cap \overline{C}$ *is it also an element of* Y*?*
(b) Draw separate Venn diagrams for X *and* Y *and conjecture whether either set is a subset of the other.*
(c) Give an example of actual sets A*,* B *and* C *for which* $X = Y$*.*

Problem 10. *For each* $n \in \mathbb{N}$ *let* A_n *be the half-open interval* $[0, 1 + 1/n)$*. Describe each of the following sets as an interval or product of intervals:*
(a) $\bigcup_{n=1}^{\infty} A_n$ *(b)* $\bigcap_{n \in \mathbb{N}} A_n$ *(c)* $\bigcap_{n \in \mathbb{N}} [0, 2] \times A_n$*.*

Problem 11. *Consider the statement:*

Let a*,* m *and* n *be integers. If* m *divides* a *and* n *divides* a *then* mn *divides* a*.*

Decide whether or not this statement is true and provide convincing justification to support your answer.

Problem 1. *Give an elementwise proof that* $(A - B) \cap (A - C)$ *is a subset of* $\overline{B}$ *for all sets* A, B *and* C.

ANSWER: Let A, B and C be sets. Suppose that x is an element of $(A - B) \cap (A - C)$. By the definition of intersection this means that $x \in A - B$ and $x \in A - C$. In particular x is an element of $A - B$ and therefore $x \in A$ and $x \notin B$ by the definition of set difference. Since $x \notin B$, we conclude that $x \in \overline{B}$ by the definition the complement of a set. Therefore, each element x in the set $(A - B) \cap (A - C)$ is also an element of the set $\overline{B}$ and this implies that $(A - B) \cap (A - C) \subseteq \overline{B}$ by the definition of subset. □

Problem 2. *Consider the proposition 'The square of an even number is an even number'.*
(a) Describe this proposition as an implication statement.
(b) Give the contrapositive statement..
(c) Give the converse statement.
(d) Which of (a),(b), (c) are true and which are false.

ANSWER:
(a) The statement can be rephrased as "*If an integer is even then its square is even*". This has the form of an implication statement "$p \implies q$" where, for an integer n, p is the statement "*n is even*" and q is the statement "*n^2 is even*".
(b) "*If the square of an integer is odd then the integer is odd*".
(c) "*If the square of an integer is even then the integer is even*".
(d) All of the statements are true.

Problem 3. *Use a proof by contradiction to show that if* A, B *and* C *are sets and* A *is a subset of* C *then*

$$A \cap \overline{(B \cup C)} = \emptyset \ .$$

ANSWER: Let A, B and C be sets and suppose that A is a subset of C. Assume for contradiction that $A \cap \overline{(B \cup C)} \neq \emptyset$. By the definition of empty set this means that there is at least one element in the set $A \cap \overline{(B \cup C)}$. Let x be such an element. Since $x \in A \cap \overline{(B \cup C)}$, we know that (1) $x \in A$ and (2) $x \in \overline{B \cup C}$. By the definition of the complement of a set, condition (2) is equivalent to saying that x is not an element of $B \cup C$. Because $B \cup C$ consists of all elements which are in B or C, $x \notin B \cup C$ means that x is neither an element of B nor an element of C, that is $x \notin B$ and $x \notin C$ On the other hand since A is a subset of C (by hypothesis) and $x \in A$ (by (1)), it must be true that $x \in C$. So we have shown that $x \notin C$ and $x \in C$ and this is a contradiction. We conclude that the assumption $A \cap \overline{(B \cup C)} \neq \emptyset$ must be false, and thus $A \cap \overline{(B \cup C)}$ equals the empty set. □

Problem 4. *Is it true for all sets* A, B *and* C *that* $\overline{(\overline{C} \cup (A \cap B))}$ *is a subset of* $A \cap C$*? Either prove your assertion or give a counterexample.*

ANSWER: A careful analysis of the Venn diagrams for $\overline{(\overline{C} \cup (A \cap B))}$ and $A \cap C$ should convince you that the given statement false. Believing that the statement is false suggests that we provide a counterexample. Choosing sets A, B and C by $A = B = \emptyset$ and $C = \{1\}$ results in $\overline{(\overline{C} \cup (A \cap B))} = \{1\}$ and $A \cap C = \emptyset$. This clearly demonstrates that the stated inclusion is not true for all sets A, B, and C.
COMMENT: A more careful look at the Venn diagrams will actually show that the stated inclusion <u>IS</u> true if and only if the sets A, B and C are chosen so that C is a subset of A. Notice that the given counterexample does not satisfy this condition. So any example consisting of three sets A, B and C where $C \subseteq A$ would serve as a counterexample to the statement.

Problem 5. *Use an elementwise approach to prove the statement: For all sets* A, B *and* C, $(B \cup C) - A = (B - A) \cup (C - A)$.

ANSWER COMMENT: Since this statement asserts that two sets are equal, two elementwise proofs will be needed in order to establish that each set is a subset of the other...

Problem 6. *Determine if each statement is true or false. Provide justification as appropriate.*
(a) $\{a\} \in \{\{a\}\}$ *TRUE. In fact* $\{a\}$ *is the only element in the set* $\{\{a\}\}$.
(b) $\mathbb{Z} \in \mathbb{Z}$ *FALSE. Since* $\mathbb{Z}$ *is a set it cannot be an element of* $\mathbb{Z}$, *as every element of* $\mathbb{Z}$ *is a number.*
(c) $\mathbb{Z} \subseteq \mathbb{Z}$. *TRUE. Every set is a subset of itself.*
(d) $\varnothing \subseteq \{\varnothing\}$. *TRUE. The empty set is a subset of every set.*
(e) $\{\varnothing\} \subseteq \varnothing$. *FALSE.* $\{\varnothing\}$ *has an element (namely* $\varnothing$*) whereas* $\varnothing$ *contains no elements.*
(f) $\{\{\{\varnothing\}\}\} \subseteq \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$. *TRUE. The set* $\{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}$ *has 3 elements one of which is* $\{\{\varnothing\}\}$.
(g) The set $\{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$ *has cardinality 4.* *FALSE. The cardinality of this set is* 3.
(h) If P *and* Q *are statements then* $\neg(P \implies Q)$ *is logically equivalent to* $\neg P \implies \neg Q$.
FALSE. This can be shown by writing out the truth tables for $\neg(P \implies Q)$ *and* $\neg P \implies \neg Q$.

Problem 7. *Let* A *and* B *be sets. State the negation of each of the following propositions as directly as possible.*
(a) B *is a subset of* A *or* $A \cap B = \emptyset$.
(b) If B *is a subset of* A *then* A *and* B *are disjoint.*
(c) For every x *in* A, B *contains an element other than* x.

ANSWER:
(a) First note that if a statement of the form "$\mathcal{P}$ or $\mathcal{Q}$" is true then either $\mathcal{P}$ is true or $\mathcal{Q}$ is true (or possibly both). So if "$\mathcal{P}$ or $\mathcal{Q}$" is false then $\mathcal{P}$ and $\mathcal{Q}$ must both be false.
So a correct answer to (a) is: B *is not a subset of* A *and* $A \cap B \neq \emptyset$.
(b) First note that if a statement of the form "if $\mathcal{P}$ then $\mathcal{Q}$" is false then it must be that $\mathcal{P}$ is false and $\mathcal{Q}$ is true. Therefore a correct answer to (b) is: B *is a subset of* A, *and* A *and* B *are not disjoint.*
(c) There exists an element x in A such that every element of B equals x. (In other words, B is a one-element subset of A.)

Problem 8. *Let* A *be set whose power set* $\mathcal{P}(A)$ *has* 32 *elements.*
(a) What is the cardinality of A*?*
(b) Does $\mathcal{P}(\mathcal{P}(A))$ *have more than a million elements? Explain.*

ANSWER: (a) Since $|\mathcal{P}(A)| = 2^{|A|}$ and $32 = 2^5$, the set A must have 5 elements in it.
(b) The cardinality of $\mathcal{P}(\mathcal{P}(A))$ is $2^{|\mathcal{P}(A)|} = 2^{32}$ and this is larger than $1,000,000$ since

$$2^{32} = (2^4)^8 = 16^8 > 10^8 > 10^6 = 1,000,000.$$

Problem 9. *For sets* A, B *and* C *let*

$$X = A \cap \big((B \cup C) - (B \cap C)\big) \cap B \quad \text{and} \quad Y = (B - C) \cap A.$$

(a) If x *is an element of* $A \cap B \cap \overline{C}$ *is it also an element of* Y*?*
(b) Draw separate Venn diagrams for X *and* Y *and conjecture whether either set is a subset of the other.*
(c) Give an example of actual sets A, B *and* C *for which* $X = Y$.

ANSWER:
(a) Yes, because $x \in A$ and $x \in B - C = \{t \mid t \in B \text{ and } t \notin C\}$.
(b) In fact the sets X and Y are equal–so $X \subseteq Y$ and $Y \subseteq X$.
(c) Since $X = Y$ any choice of sets A, B and C would work. The most basic one would be $A = B = C = \varnothing$.
(COMMENT: I didn't check this problem carefully enough ahead of time. I had meant to give one where $X \subseteq Y$ but $Y \not\subseteq X$, and then the example in (c) would need to be chosen more carefully.)

Problem 10. *For each* $n \in \mathbb{N}$ *let* A_n *be the half-open interval* $[0, 1 + 1/n)$. *Describe each of the following sets as an interval or product of intervals:* *(a)* $\bigcup_{n=1}^{\infty} A_n$ *(b)* $\bigcap_{n\in\mathbb{N}} A_n$ *(c)* $\bigcap_{n\in\mathbb{N}} [0, 2] \times A_n$.

ANSWER: It's a good idea to first get a sense of the pattern of the subsets A_n. We have

$$A_1 = [0,2), \quad A_2 = [0,3/2), \quad A_3 = [0,4/3), \quad A_4 = [0,5/4), \quad A_5 = [0,6/5), \ldots$$

and this would be referred to as a 'nested, decreasing family of sets'. Notice that the numbers $1 + 1/n$ are $2, 3/2, 4/3, 5/4, 6/5, \ldots$, and this forms a decreasing sequence whose limit equals 1 (approaching 1 from the right).
(a) $\bigcup_{n=1}^{\infty} A_n = A_1 = [0,2)$ because $A_n \subseteq A_1$ for each natural number n.
(b) $\bigcap_{n \in \mathbb{N}} A_n = [0,1]$. This is because the closed interval $[0,1]$ is a subset of A_n for each $n \in \mathbb{N}$ and the numbers $1 + 1/n$ limit to 1 from the right.
(c) $\bigcap_{n \in \mathbb{N}} [0,2] \times A_n = [0,2] \times [0,1] = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1\}$

Problem 11. *Consider the statement:*
Let a, m and n be integers. If m divides a and n divides a then mn divides a.
Decide whether or not this statement is true and provide convincing justification to support your answer.

ANSWER: The statement is false. One example might be given by taking $a = m = n = 3$ then $m = 3$ and $n = 3$ divide $a = 3$, but $mn = 9$ does not divide $a = 3$. (COMMENT: You might observe that it **is** true that: If m divides a and n divides a then mn divides a^2.)