Math 2513
Creating elementwise proofs in set theory.

Let $\mathcal{P}$ and $\mathcal{Q}$ be statements, and suppose that we want to verify the implication statement $\mathcal{P} \implies \mathcal{Q}$. The standard thought process (or "scratch work") for constructing a direct proof would typically go as follows:

STEP 1: Understand the Problem. Clearly identify the two statements $\mathcal{P}$ and $\mathcal{Q}$, and be certain that you understand the basic definitions of all of the terms that occur. In some circumstances you may also want to remind yourself of some basic principles that have previously been proved regarding these definitions.
STEP 2: Examine the Hypothesis. Analyze the statement $\mathcal{P}$ using definitions to reinterpret what it says in the most elementary terms possible.
STEP 3: Examine the Conclusion. Analyze the statement $\mathcal{Q}$ using definitions to reinterpret what it says in the most elementary terms possible.
STEP 4: Bridge the gap. Examine the reinterpretations of $\mathcal{P}$ and $\mathcal{Q}$ from steps 2 and 3, and see if you can explain how to fill in the gap going from one to the other. Of course, some proofs may be very intricate, and this step may be difficult. But, on the other hand, many times it will be straight forward and then you will have successfully outlined a proof.

After completing this scratch work, you will now be ready to write a formal proof of the statement $\mathcal{P} \implies \mathcal{Q}$ (starting with "Proof" and ending with $\Box$). The sentences in your proof should break into five parts.

Part 1: Write sentences to declare any variables that are required, and to assume that the hypothesis statement $\mathcal{P}$ is true.
Part 2: Write sentences from STEP 2 of your scratch work which explain the meaning of $\mathcal{P}$.
Part 3: Write sentences from STEP 4 bridging the gap from your reinterpretation of $\mathcal{P}$ to your reinterpretation of $\mathcal{Q}$.
Part 4: Write sentences explaining how the reinterpretation of $\mathcal{Q}$ implies $\mathcal{Q}$ itself. This will usually consist of reversing the analysis from STEP 3 of your scratch work.
Part 5: Summarize your final conclusion as necessary.

Here's an illustration.

Example: Use an elementwise proof to show that $A-(B\cup C)$ is a subset of $(A-B)\cup C$ for all sets A, B and C.

Scratch Work:
STEP 1: To show that one set is a subset of another set we need to verify that each element of the first set is also an element of the second set. Here the given statement has the form: "*If x is an element of $A-(B\cup C)$ then x is an element of $(A-B)\cup C$*". So we should interpret $\mathcal{P}$ as the statement "$x \in A-(B\cup C)$", and we should interpret $\mathcal{Q}$ as the statement "$x \in (A-B)\cup C$". For this proof we need to understand the definitions of set difference and union, each of which occur twice.

(REMEMBER: the ultimate goal is to start by assuming that x is an element of $A-(B\cup C)$ and to then logically deduce that $x \in (A-B)\cup C$.)

STEP 2 (consider $\mathcal{P}$): By the definition of set difference, the meaning of $x \in A-(B\cup C)$ is that $x \in A$ and $x \notin B\cup C$. Using the definition of union, the second condition implies that it is NOT true that $x \in B$ or $x \in C$, and we note that this can rephrased as asserting that $x \notin B$ and $x \notin C$. (By DeMorgan's Principle, the negation of "or" is "and" and vice-versa.) Altogether we have now reinterpreted the statement $x \in A-(B\cup C)$ as being the same as writing $x \in A$ and $x \notin B$ and $x \notin C$.

STEP 3 (consider $\mathcal{Q}$): By the definition of union, $x \in (A-B)\cup C$ means that $x \in A-B$ or $x \in C$. With the definition of set difference, the first of these conditions implies that $x \in A$ and $x \notin B$. This shows $x \in (A-B)\cup C$ can be reinterpreted as $x \in A$ and $x \notin B$, or $x \in C$.

STEP 4 (bridge the gap): Now we just need to observe that if $x \in A$ and $x \notin B$ and $x \notin C$ (which is the reinterpretation of $\mathcal{P}$) then $x \in A$ and $x \notin B$. From this it follows that it is true that $x \in A$ and $x \notin B$ or $x \in C$ (which is the reinterpretation of $\mathcal{Q}$).

Now we write the final proof:

Proof. *[part 1]* Let A, B and C be sets. Assume that $x \in A-(B\cup C)$. *[part 2]* By the definition of set difference, it follows that $x \in A$ and $x \notin B\cup C$. Using the definition of union, it is NOT true that $x \in B$ or $x \in C$, and, by DeMorgan's Principle, this can rephrased as $x \notin B$ and $x \notin C$. *[part 3]* From these observations we observe that $x \in A$ and $x \notin B$. Therefore it is true that $x \in A$ and $x \notin B$, or $x \in C$. *[part 4]* As a consequence we observe that $x \in A-B$ (by the definition of set difference) or $x \in C$, and that $x \in (A-B)\cup C$ (by the definition of union). *[part 5]* We have shown that each element x of $A-(B\cup C)$ is also an element of $(A-B)\cup C$ which completes the proof (using the definition of subset). $\square$

(Of course we wouldn't actually include "*[part 1]*", etc.)