Math 2513: ## About "Element-wise Proofs" in Set Theory

An "element-wise proof" is a method for showing that one set is a subset of another set. This is the most convincing technique to use for proving subset inclusion. The basic strategy of the element-wise technique is to start with an arbitrary element of the first set, and then to verify that that element must also be in the second set, using basic definitions and commonplace deductive logic.

Notice that element-wise proofs can also be used to show that two sets A and B are equal since $A = B$ is equivalent to asserting that $A \subseteq B$ and $B \subseteq A$. In this situation you would actually need two element-wise proofs, the first showing that each element of A is in B and the second showing that each element of B is in A.

Here are two illustrations of the elementwise method.

THEOREM 1. *For all sets A, B and C, $(A \cap B) - C$ is a subset of $((A \cup B) - C) \cap (A - C)$.*

Proof. Let A, B and C be sets. Let x be an element of $(A \cap B) - C$. By the definition of set difference this means that $x \in A \cap B$ but $x \notin C$. Using the definition of intersection, it follows that $x \in A$ and $x \in B$ but $x \notin C$. Since $x \in A$, it is true that x is an element of A or B, and so $x \in A \cup B$ (by the definition of union). As $x \in A \cup B$ and $x \notin C$, we see that $x \in (A \cup B) - C$ using the definition of set difference. The same definition also applies to show that $x \in A - C$ because $x \in A$ and $x \notin C$. So we have verified that $x \in (A \cup B) - C$ and $x \in A - C$, which means that x is in the intersection of these two sets (by the definition of intersection). To summarize, it has been shown that each element of $(A \cap B) - C$ is also an element of $((A \cup B) - C) \cap (A - C)$, and so $(A \cap B) - C \subseteq ((A \cup B) - C) \cap (A - C)$ by the definition of subset. □

COMMENTS:
(1) Notice how the first sentence of the proof states clearly what the variables A, B and C stand for. Every proof should start off with a similar statement of "declaration of variables".
(2) Often it is convenient to introduce new variables in the course of a proof in order to make it easier to explain to the reader what you are doing. The second sentence of this proof "*Let x be an element of* $(A \cap B) - C$" is an example of this. It is important to realize that this will fix the meaning of the variable "x" once and for all throughout the entire proof (or until you declare otherwise). Since we haven't specified anything about x other than that it is an element of $(A \cap B) - C$, this really means that it represents an arbitrary element of the set. The strategy of the element-wise proof is to now go on and show that x is also an element of the second set (which is $((A \cup B) - C) \cap (A - C)$ in this case) using basic principles of (common sense) logic.
(3) It is not hard to see that the two sets in this theorem $(A \cap B) - C$ and $((A \cup B) - C) \cap (A - C)$ need not be equal. One example that illustrates this is obtained by taking $A = \{a\}$ (the set having a single element a) and taking $B = C = \emptyset$. With these choices, it can be observed that $(A \cap B) - C = \emptyset$ while $((A \cup B) - C) \cap (A - C) = \{a\}$, and these two sets are not equal. This specific choice of sets would be referred to as a "counterexample" showing that $((A \cup B) - C) \cap (A - C)$ is not a subset of $(A \cap B) - C$ for all sets A, B and C.

THEOREM 2. *For all sets A, B and C, if A is a subset of B and B is a subset of C then A is a subset of C.*

Proof. Let A, B and C be sets, and assume that $A \subseteq B$ and $B \subseteq C$. Let t be an element of A. Since $A \subseteq B$ and $t \in A$ then t must be an element of B by the definition of "subset". Since $B \subseteq C$ and $t \in B$ then t must be an element of C by the definition of "subset". Therefore we have shown that each element t of A is also an element of C. By the definition of "subset" this shows that A is a subset of C. □

COMMENT: The statement to be proved here is an if/then statement (or implication), and this means that it has a hypothesis (or hypotheses). In the proof of an if/then statement, after the declaration of variables is completed you should carefully state what the hypotheses are. Here this was achieved in the first sentence by writing "*assume that $A \subseteq B$ and $B \subseteq C$*".