Discrete Math Group Project #10
10/30/20 (due by 6PM on Wednesday, 11/04)

PART I: Mathmatical Induction.

PROBLEM #1. Use induction to prove:

> **Theorem.** For each integer $n \geq 1$ we have $1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6}$.

Model your wording on the proof given on page 5 of *Writing Proofs* by Christopher Heil (posted at the course web site).

Hammack discusses mathematical induction in Chapter 10 of his book. His discussions on pages 180-186 provide more context for how and why the technique works, and you should read through this also.

PART II: Recursively Defined Sequences.

PROBLEM #2. Let $a_0 = 1$ and assume that for each integer $n \geq 1$, a_n satisfies

$$a_n = 5a_{n-1} + 3. \tag{1}$$

(a) Write out the first 5 terms of the sequence $(a_n) = \{a_n \mid n \in \mathbb{Z}_{\geq 0}\}$.

(b) Use mathematical induction to show that $a_n = 5^n + \dfrac{3}{4}(5^n - 1)$ for all $n \in \mathbb{Z}_{\geq 0}$.

A sequence (a_n) satisfying an equation like that in (1) is said to be defined recursively. More generally, a sequence (a_n) is defined recursively if the term a_n is defined as a function of the preceding terms $\{a_0, a_1, \ldots, a_{n-1}\}$. A well-known example is the Fibonacci sequence in which $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. We say that the Fibonacci sequence is defined by a "two-step" recursion because the general nth term is determined by the two preceding terms. On the other hand, the recursion described in (1) is an example of a one-step recursion.

PROBLEM #3. Assume that a_n satisfies the 2-step recursion

$$a_n = 2a_{n-2} + 3a_{n-1} \tag{2}$$

for $n \geq 2$. Use mathematical induction to show that $a_n = \left(\frac{3+\sqrt{17}}{2}\right)^n$ and that $a_n = \left(\frac{3-\sqrt{17}}{2}\right)^n$ are solutions to the recursion.