Exam 2 Answers

#1. (a) $f: X \to Y$ is surjective iff for each $y \in Y$ there is $x \in X$ with $f(x) = y$.

(b) $f: X \to Y$ is not injective iff there are elements $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$.

#2. $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ with $f(m,n) = (m+n, m+3n)$ is injective.

<u>proof</u> Let (m_1, n_1) and (m_2, n_2) be elements of $\mathbb{Z}\times\mathbb{Z}$ with $f(m_1, n_1) = f(m_2, n_2)$. This means that

$$(m_1+n_1, m_1+3n_1) = (m_2+n_2, m_2+3n_2).$$

From this we see that the integers m_1, n_1, m_2 and n_2 satisfy the two equations:

$$(1) \quad m_1 + n_1 = m_2 + n_2 \quad , \text{ and}$$

$$(2) \quad m_1 + 3n_1 = m_2 + 3n_2$$

By subracting equation (1) from equation (2) we get

$$2n_1 = (m_1+3n_1) - (m_1+n_1) = (m_2+3n_2) - (m_2+n_2) = 2n_2$$

and dividing both sides of this equation by 2 shows that $n_1 = n_2$. Therefore

$$m_1 = (m_1+n_1) - n_1 = (m_1+n_1) - n_2 = (m_2+n_2) - n_2 = m_2$$

and $(m_1, n_1) = (m_2, n_2)$. We have shown that $(m_1, n_1) = (m_2, n_2)$ whenever $f(m_1, n_1) = f(m_2, n_2)$ which verifies that the function f is injective. □

#3. (a) If n is even then $(-1)^{n+1} = -1$ and $f(n) = n-1$. If n is odd then $(-1)^{n+1} = 1$ and $f(n) = n+1$. Thus

$$f(n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$$

(b) $f(2022) = 2022 - 1 = 2021$ since 2022 is even.

(c) No. $f(2) = 1$ and $f(2021) = 2022$ so $1, 2022 \in f([1, 2021])$. But $2021 \notin f([1, 2021])$ because if $f(n) = 2021$ then n must be even and equal to 2022. This shows that $2021 \notin f([1, 2021])$ and that $f([1, 2021])$ is not an interval in $\mathbb{N}$. In fact it can be observed that

$$f([1, 2021]) = ([1, 2020] \cap \mathbb{N}) \cup \{2022\}$$

which is not an interval.

(d) <u>Yes</u>. If $n \in \mathbb{N}$ is even then $n-1$ is odd and $f(n-1) = (n-1)+1 = n$. If $n \in \mathbb{N}$ is odd then $n+1$ is even and $f(n+1) = (n+1) - 1 = n$.

#4. (a) In $\mathbb{Z}_{15}$, $-7 = 8$ (because $7 + 8 = 15 = 1\times 15 + 0$).

(b) In $\mathbb{Z}_{15}$, $7 \cdot 13 = 1$ (b/c $7 \times 13 = 91 = 6 \times 15 + 1$).

(c) Suppose that $7x + 4 = 0$ in $\mathbb{Z}_{15}$. Then $7x = -4$ and multiplying both sides by 13 shows that

$$x = 1 \cdot x = (13 \cdot 7) \cdot x = 13(7x) = 13(-4) = (-13)(4)$$
$$= (2)(4) = 8$$

(d) The solution to $7x + b = 0$ in $\mathbb{Z}_{15}$ is

$$x = (-13) \cdot (b) = 2 \cdot b$$

<u>#5</u> (a) The number of 4 element subsets of a set with 9 elements is $\binom{9}{4} = \frac{9!}{4!\,5!} = \frac{9\cdot 8\cdot 7\cdot 6}{4\cdot 3\cdot 2\cdot 1} = 126$

(b) $\binom{8}{3} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = 56$

(c) The number of subsets containing both 5 and 6 is $\binom{7}{2} = 21$. So there are $126 - 21 = 105$ 4-element subsets that don't contain both 5 and 6.

<u>#6</u> (a) $2^{10} = 1024$ using product principle 9 times... Choose a bit string in 10 steps...

(b) $\binom{10}{5} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 252$

(c) $\text{range}(F) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

For example, $F(1111111111) = 0$ and $F(0000000000) =$

(d) There are 35 elements $w \in W$ with $F(w) = 4$.

One approach to counting these is to break into separate cases depending on the length $N(w)$ of the longest string of consecutive 0's in the bit string w. If $F(w) = 4$ then $N(w)$ must be between 4 and 6.

<u>$N(w) = 6$</u> There are 26 of these.

<u>$N(w) = 5$</u> There are 6 of these

<u>$N(w) = 4$</u> There are 3 of these