1. How many 6 letter combinations with repetition can made using the letters $\{a,b,c,d\} = X$?

So $|X| = 4 = n$ and $k = 6$

some examples:

aabdca ← equal as combinations

" badaca

" aaabcd

$\#a = 3$ $\#c = 1$
$\#b = 1$ $\#d = 1$

To solve the problem we must indicate the # of a's, # of b's, # of c's, # of d's.

Consider strings of $\ast$'s and $|$'s (stars and bars) with 3 bars and 6 stars.

compartment 2, compartment 4

$\underbrace{\ast\ast\ast}_{\text{compartment } 1}|\ast|\ast|\ast$ ← string of length 9 with 3 bars and 6 stars.

compartment 3

$\overset{a}{\ast\ast\ast}|\overset{b}{\ast}|\overset{c}{\ast}|\overset{d}{\ast}$ ⇝ encodes that we have 3 a's, 1 b, 1 c, 1 d.

$|\ast\ast\ast|\ast|\ast\ast$ ← 0 a's, 3 b's, 1 c, 1 d
a b c d

How many strings of length 9 with 3 bars and 6 stars are there? aaccd

$\ast$ $\ast$ $|$ $|$ $\ast$ $\ast$ $|$ $\ast$ $\ast$
a b c d

Out of a possible slots choose a set of 3 slots in which to put bars.

There are $\binom{9}{3} = \frac{9!}{6!\,3!} = \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} = 84$

$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

2. How many k letter combinations with repetition can be made choosing letters from an n-element set?

Consider strings with $n-1$ bars and with k stars. These strings have length $n-1+k$.

From $n-1+k$ slots choose $n-1$ positions to put the bars. The number of ways to do this is $\binom{n+k-1}{n-1}$.

3. How many ways can 20 be written as a sum of 5 non-negative integers $20 = n_1 + n_2 + n_3 + n_4 + n_5$?

$$n_1 + n_2 + n_3 + n_4 + n_5 = 20 \qquad (\ast)$$

Consider strings with 4 bars and 20 stars

examples $6 + 1 + 0 + 1 + 12 = 20$

$\underset{6}{\ast\ast\ast\ast\ast\ast}|\underset{1}{\ast}|\underset{0}{}|\underset{1}{\ast}|\underset{12}{\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast}$

Answer There are $\binom{24}{4} = \frac{24!}{4!\,20!} = \frac{24\cdot 23\cdot 22\cdot 21}{4\cdot 3\cdot 2\cdot 1} = 10{,}626$

$1 + 0 + 6 + 12 + 1 = 20$

$\ast||\ast\ast\ast\ast\ast\ast|\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast|\ast$

4. How many ways can we write $20 = n_1 + n_2 + n_3 + n_4 + n_5$ if each $n_k \in \mathbb{N}$? (Not allowing 0's)

$$n_1 + n_2 + n_3 + n_4 + n_5 = 20$$

$$(n_1 - 1) + (n_2 - 1) + (n_3 - 1) + (n_4 - 1) + (n_5 - 1) = 15$$

Observe that each $n_k - 1 \geq 0$.

Answer Consider strings with 15 $\ast$'s and 4 bars.

There are $\binom{19}{4} = \frac{19!}{4!\,15!} = 3{,}876$