Class Problem
Math 2513
Monday, June 27

PROBLEM. Use the principle of mathematical induction to prove that

$$3 + 11 + 19 + \cdots + (8n - 5) = 4n^2 - n \tag{1}$$

for all positive integers n. Before starting on your proof be sure to describe the statement $\mathcal{P}(n)$ that you will work very carefully.

Comment. The sum on the left hand side of equation (1) can be expressed as $\sum_{k=1}^{n} 8k - 5$ using summation notation. You don't need to use this here but in the future we may use this notation to write a formula like this.

SOLUTON:

Proof. For each positive integer n let $\mathcal{P}(n)$ be the statement

$$\text{"}3 + 11 + 19 + \cdots + (8n - 5) = 4n^2 - n\text{"}.$$

Basis Step: The LHS (left-hand side) of the statement $\mathcal{P}(1)$ is $3 + 11 + \cdots + (8(1) - 5) = 3$. The RHS (right-hand side) of $\mathcal{P}(1)$ is $4(1)^2 - 1 = 4 - 1 = 3$. Since these are equal this shows that $\mathcal{P}(1)$ is true. This completes the proof of the basis step.

Inductive Step: Let k be a positive integer and assume that $\mathcal{P}(k)$ is true. This means that

$$3 + 11 + 19 + \cdots + (8k - 5) = 4k^2 - k.$$

The LHS of $\mathcal{P}(k + 1)$ is

$$3 + 11 + 19 + \cdots + (8(k + 1) - 5) = \big(3 + 11 + 19 + \cdots + 8k\big) + (8(k + 1) - 5),$$

and since $\mathcal{P}(k)$ is true we can write this as

$$\big(4k^2 - k\big) + (8(k + 1) - 5) = 4k^2 + 7k + 3.$$

The RHS of $\mathcal{P}(k + 1)$ is

$$4(k + 1)^2 - (k + 1) = 4(k^2 + 2k + 1) - k - 1 = 4k^2 + 7k + 3.$$

Therefore $\mathcal{P}(k + 1)$ is true since its left and right hand sides are equal. This shows (by a direct proof) that $\mathcal{P}(k)$ implies $\mathcal{P}(k + 1)$, and completes the proof of the inductive step.

We conclude that $\mathcal{P}(n)$ is true for every positive integer n by the Principle of Mathematical Induction.

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