Problem. Use the principle of mathematical induction to prove that
\[ 3 + 11 + 19 + \cdots + (8n - 5) = 4n^2 - n \]  
for all positive integers \( n \). Before starting on your proof be sure to describe the statement \( P(n) \) that you will work very carefully.

Comment. The sum on the left hand side of equation (1) can be expressed as \( \sum_{k=1}^{n} 8k - 5 \) using summation notation. You don’t need to use this here but in the future we may use this notation to write a formula like this.

SOLUTION:

Proof. For each positive integer \( n \) let \( P(n) \) be the statement

\[ "3 + 11 + 19 + \cdots + (8n - 5) = 4n^2 - n". \]

Basis Step: The LHS (left-hand side) of the statement \( P(1) \) is \( 3 + 11 + \cdots + (8(1) - 5) = 3 \). The RHS (right-hand side) of \( P(1) \) is \( 4(1)^2 - 1 = 4 - 1 = 3 \). Since these are equal this shows that \( P(1) \) is true. This completes the proof of the basis step.

Inductive Step: Let \( k \) be a positive integer and assume that \( P(k) \) is true. This means that

\[ 3 + 11 + 19 + \cdots + (8k - 5) = 4k^2 - k. \]

The LHS of \( P(k + 1) \) is

\[ 3 + 11 + 19 + \cdots + (8(k + 1) - 5) = (3 + 11 + 19 + \cdots + 8k) + (8(k + 1) - 5), \]

and since \( P(k) \) is true we can write this as

\[ (4k^2 - k) + (8(k + 1) - 5) = 4k^2 + 7k + 3. \]

The RHS of \( P(k + 1) \) is

\[ 4(k + 1)^2 - (k + 1) = 4(k^2 + 2k + 1) - k - 1 = 4k^2 + 7k + 3. \]

Therefore \( P(k + 1) \) is true since its left and right hand sides are equal. This shows (by a direct proof) that \( P(k) \) implies \( P(k + 1) \), and completes the proof of the inductive step.

We conclude that \( P(n) \) is true for every positive integer \( n \) by the Principle of Mathematical Induction.