Problem. Consider the two functions $f$ and $g$ which are defined by:

\[ f : \mathbb{R} \to (0, \infty) \text{ where } f(x) = \frac{1}{x^2+1} \text{ for each real number } x, \text{ and} \]

\[ g : (0, \infty) \to (0, \infty) \text{ where } g(t) = \frac{1}{t^2+1} \text{ for all positive real numbers } t. \]

(1) Explain carefully why $f$ and $g$ are different functions.
(2) Determine the range of $f$.
(3) Show that the function $f$ is not one-to-one.
(4) Show that the function $g$ is one-to-one.

Notes: (a) Write in sentences please. (b) In the definition of the functions, $(0, \infty)$ represents the open interval from 0 to $+\infty$ in the real line; this coincides with the set of positive real numbers.

(1) The domain of $f$ is the set $\text{domain}(f) = \mathbb{R}$ of real numbers while the domain of $g$ is the set $\text{domain}(g) = (0, \infty)$ of positive real numbers. Since these two sets are not equal, the two functions are different. (The domain is part of the definition of a function, as is the codomain also.)

(2) The range of $f$ is $f(\mathbb{R}) = (0, 1] = \{ t \in \mathbb{R} \mid 0 < t \leq 1 \}$. (As a real-valued function, $f$ is an even function, it has an absolute maximum at $x = 0$, it is decreasing on the interval $(0, \infty)$ and $\lim_{x \to +\infty} f(x) = 0$. This example is a good reminder of how difficult it can be to determine the range of a function.)

(3) As an example, $\sqrt{5}$ and $-\sqrt{5}$ are two distinct elements in the domain of $f$ (they are both real numbers). Since $f(\sqrt{5})$ and $f(-\sqrt{5})$ both equal $1/26$, this shows that $f$ is not a one-to-one function.

(4) Let $s$ and $t$ be elements of $(0, \infty)$ for which $g(s)$ equals $g(t)$. Then $\frac{1}{1+s^2} = \frac{1}{1+t^2}$. Therefore $1+s^2 = 1+t^2$ which implies that $t = \pm s$. Since both $s$ and $t$ are positive, $t$ cannot equal $-s$ (since $s$ is positive, $-s$ must be negative). Therefore $t$ must equal $s$. We have shown that if $s, t \in (0, \infty)$ and $g(s) = g(t)$ then $s = t$. By the definition of one-to-one it follows that $g$ is one-to-one.