EXAM 1—Solutions
Math 2513
9-17-04

NOTE: Many of the problems on this test can be solved in more than one way.

1. Consider the set identity $A - B \subseteq A$.
(a) Write down complete statements of the two definitions which play central roles in this identity.
(b) Prove the identity.

(a) The two relevant definitions are "set difference" and "subset". Let A and B be sets. The *set difference* $A - B$ is defined to be the set $\{x \mid x \in A \text{ and } x \notin B\}$. To say that A *is a subset of* B means that whenever $x \in A$ then $x \in B$.

(b) Let A and B be sets. Suppose that $x \in A - B$. By the definition of set difference this means that $x \in A$ and $x \notin B$. In particular, $x \in A$. So we have shown that each element x in $A - B$ is also an element of A which means that $A - B \subseteq A$. $\square$

2. In one or two sentences describe the standard general strategy which you would use to prove that two given sets are equal.

To prove that two sets are equal the standard strategy is to show that each is a subset of the other. So, if the sets to be shown equal are A and B then the proof breaks into two steps (Step 1) $A \subseteq B$ and (Step 2) $B \subseteq A$.

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function described by the rule $f(n) = n^2 + 1$. (a) What are the domain, codomain and range of this function f? (b) Show (by example) that f is not injective. (c) Determine $f(S)$ if $S = \{-1, 0, 1, 2, 3\}$.

(a) $domain(f) = \mathbb{Z}$, $codomain(f) = \mathbb{Z}$, $range(f) = \{n^2 + 1 \mid n \in \mathbb{Z}\} = \{1, 2, 5, 10, 17, 26, 37, \dots\}$ (Note that the range of f is a proper subset of $\mathbb{Z}^+$.)
(b) $f(3) = 3^2 + 1 = 10$ and $f(-3) = (-3)^2 + 1 = 10$. So $f(3) = f(-3)$ but $3 \neq -3$, and this shows that f is not injective.
(c) $f(\{-1, 0, 1, 2, 3\}) = \{1, 2, 5, 10\}$.

4. Let A and B be sets such that $A \subseteq B$. Show that $A \cup B \subseteq B$.

Let A and B be sets such that $A \subseteq B$. Suppose that $x \in A \cup B$. By the definition of union, (i) $x \in A$ or (ii) $x \in B$. If (i) holds then $x \in B$ using the hypothesis that $A \subseteq B$ and the definition of subset. So in either case (i) or (ii) we can conclude that $x \in B$. Thus $x \in A \cup B$ implies that $x \in B$, and this shows that $A \cup B \subseteq B$ by the definition of subset. $\square$

5. Let A and B be two non-empty sets. (a) Describe the Cartesian product $A \times B$. (b) Let $P : A \times B \to B$ be the function which maps (a, b) to b. Show that P is onto.

(a) $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.
(b) Let A and B be nonempty sets and let $P : A \times B \to B$ be the function defined by $P(a, b) = b$ for each ordered pair (a, b) in $A \times B$. Suppose that $b \in B$. Since A is non-empty there is an element $a_0 \in A$. Then

$(a_0, b) \in A \times B$ and $P(a_0, b) = b$. This shows that b is in the range of P, and P is onto. $\square$

6. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) = 3x - 2$ and $g(x) = ax + b$ where a and b are constants. (a) Determine an equation for $g \circ f(x)$ (b) Find values for a and b such that $g \circ f(x) = x$ for every $x \in \mathbb{R}$. Then determine $f \circ g(x)$ when these values of a and b are used.

(a) $g \circ f(x) = g(f(x)) = g(3x - 2) = a(3x - 2) + b = 3ax - 2a + b$
(b) If $g \circ f(x) = x$ for all $x \in \mathbb{R}$ then $3ax - 2a + b = 1x + 0$ and we must have $3a = 1$ and $-2a + b = 0$. Solving for a and b yields $a = 1/3$ and $b = 2a = 2/3$. (Another approach is to plug in two values for x and solve the resulting equations: If we choose $x = 0$ and $x = 1$ then we get equations $-2a + b = 0$ and $a + b = 1$, which yields $a = 1/3$ and $b = 2a = 2/3$.) With this choice of a and b we have

$$f \circ g(x) = f(g(x)) = f(x/3 + 2/3) = 3(x/3 + 2/3) - 2 = x.$$

(NOTE: The function g with $g(x) = x/3 + 2/3$ is the inverse function of f.)

7. Explain why the function $h : \{a, b, c, d\} \to \{1, 2, 3, 4\}$ defined by $h(a) = 2$, $h(b) = 4$, $h(c) = 3$, and $h(d) = 1$ is a bijection.

We note that a is the only element in the domain of h with $h(a) = 2$, b is the only element in $domain(h)$ with $h(b) = 4$, c is the only element in $domain(h)$ with $h(c) = 3$, and d is the only element in $domain(h)$ with $h(d) = 1$. Since a, b, c and d are all of the elements of $domain(h)$, no two distinct elements in $domain(h)$ get mapped to the same element of $\{1, 2, 3, 4\}$, and this shows that h is injective. Observe that

$$range(h) = h(\{a, b, c, d\}) = \{2, 4, 3, 1\} = codomain(h)$$

and this shows that h is surjective. Therefore h is both injective and surjective, which means that it is a bijection. $\square$

8. Let A, B and C be sets. Prove that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Let A, B and C be sets. Suppose that $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$ by the definition of union. By the definition of intersection, $x \in A$, or, $x \in B$ and $x \in C$. In either case we can see that $x \in A$ or $x \in B$, and $x \in A$ or $x \in C$. It follows that $x \in A \cup B$ and $x \in A \cup C$ (definition of union), and that $x \in (A \cup B) \cap (A \cup C)$ (definition of intersection). This shows that if $x \in A \cup (B \cap C)$ then $x \in (A \cup B) \cap (A \cup C)$ which means that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (definition of subset). $\square$