Problem like #20 page 75
Math 2513
March 8, 2005

PROBLEM. Prove that the cube of an odd number is an odd number using (a) a direct proof (b)an indirect proof (c) a proof by contradiction.

Solution: The statement to be proved is a simple implication. If we let n denote an integer then the hypothesis can be stated as "n is odd" and the conclusion is "n^3 is odd". The procedures for the three parts of the problem would be:
(a) DIRECT PROOF: *Assume that n is odd and then use logical inference to show that n^3 is odd.*
(b) INDIRECT PROOF: *Assume that n^3 is not odd and then use logical inference to show that n is not odd.* Since being not odd is equivalent to being even this can be rephrased as: *Assume that n^3 is even and then use logical inference to show that n is even.*
(c) PROOF BY CONTRADICTION: *Assume that n is odd and that n^3 is not odd and then use logical inference to derive a contradiction.* This can be rephrased as: *Assume that n is odd and that n^3 is not odd and then use logical inference to derive a contradiction.*

The key definitions here are: *An integer n is* **odd** *if there exists an integer k such that $n = 2k + 1$.*
An integer n is **even** *if there exists an integer k such that $n = 2k$.*
Note that every integer is either even or odd, and that no integer is both even and odd. (See page 63 of Rosen's text.)

(a) PROOF: Let n be an integer. Assume that n is odd. By definition of odd number this means that there is an integer k such that $n = 2k + 1$. Therefore

$$n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 \ .$$

Since $4k^3+6k^2+3k$ is an integer, this shows that n^3 is odd. This completes the proof. □

(c) PROOF: Let n be an integer. Assume that n is odd and that n^3 is even. By definition of odd number this means that there is an integer k such that $n = 2k + 1$. Therefore

$$n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 \ .$$

Since $4k^3 + 6k^2 + 3k$ is an integer, this shows that n^3 is odd but this contradicts our assumption that n^3 was even. This completes the proof by the method of proof by contradiction. □