1. (15 points) (a) Carefully state the principle of inclusion–exclusion.
(b) Illustrate part (a) by counting the number of bit strings of length 8 which contain exactly 5 ones or which start with "0110".

ANSWER:
(a) Let \( A \) be a finite set with \( A = A_1 \cup A_2 \) for two subsets \( A_1 \) and \( A_2 \) of \( A \). Then
\[
|A| = |A_1| + |A_2| - |A_1 \cap A_2|.
\]

(b) Let \( A \) be the set of bit strings of length 8 which contain exactly 5 ones or which start with "0110". Let \( A_1 \) be the set of bit strings of length 8 which contain exactly 5 ones, and let \( A_2 \) be the set of bit strings of length 8 which start with "0110". Then \( A = A_1 \cup A_2 \). Then \( |A_1| = C(8,5) = 56 \) and \( |A_2| = 2^4 = 16 \). The intersection \( A_1 \cap A_2 \) consists of bit strings of length 8 which start with "0110" and have exactly five ones, and this set has \( C(4,3) = 4 \) elements. Then, by the principle of inclusion/exclusion,
\[
|A| = |A_1| + |A_2| - |A_1 \cap A_2| = 56 + 16 - 4 = 68.
\]

2. (20 points) Use the technique of mathematical induction to prove that \( \sum_{k=2}^{n} 2k + 1 = (n+3)(n-1) \) for every integer \( n \geq 2 \).

ANSWER:
For each integer \( n \geq 2 \) let \( P(n) \) be the proposition
\[
P(n) : 5 + 7 + 9 + \cdots + (2n + 1) = (n+3)(n-1)
\]
Basis Step: The LHS of the proposition \( P(2) \) is 5 while its RHS is \((2 + 3)(2 - 1) = 5\). Since these values are equal, this shows that \( P(2) \) is true.
Inductive Step: Assume that \( P(k) \) is true for some \( k \geq 2 \). Thus \( 5 + 7 + 9 + \cdots + (2n + 1) = (k+3)(k-1) \). The LHS of \( P(k+1) \) is \( 5 + 7 + 9 + \cdots + (2k + 1) + 1 \), and we have
\[
5 + 7 + 9 + \cdots + (2k + 1) + 1 = (5 + 7 + 9 + \cdots + (2k + 1)) + (2 + 3) = (k+3)(k-1) + (2k + 3) = k^2 + 4k = (k+4)k.
\]
Since the RHS of \( P(k+1) \) is \((k+1 + 3)((k + 1) - 1) = (k + 4)k\), this shows that \( P(k+1) \) is true. Therefore we conclude that \( P(n) \) is true for all \( n \geq 2 \) using the principle of mathematical induction.

3. (10 points) Write out the entire row of Pascal's triangle whose first two entries are 1 and 9.

ANSWER:
The tenth row of Pascal's triangle consists of the ten numbers \( \binom{9}{k} \) where \( 0 \leq k \leq 9 \). The numbers are
\[
\{1, 9, 36, 84, 126, 126, 84, 36, 9, 1\}.
\]

4. (15 points) Let \( S \) be the set of all words of length 10 that can be formed using the letters \( A, B \) and \( C \) where repetition of letters is allowed.
(a) How many elements does \( S \) have?
(b) How many subsets with exactly 3 elements does \( S \) have? Show that there more than a million.
(c) Let \( S_1 \) be the subset of \( S \) consisting of words where at most two of the three letters appear. Determine \( |S_1| \).
(d) How many elements of \( S \) contain an "AABBCCAA" subword?
(e) How many elements of \( S \) contain exactly three \( A \)'s, five \( B \)'s and two \( C \)'s?
ANSWER:
(a) $|S| = 3^{10}$.
(b) $C(3^{10}, 3) = 3^{10}(3^{10} - 1)(3^{10} - 2)/6$. This number can easily be shown to be larger than $3^{24} = (27)^8 > 10^8$.
(c) $|S_1| = 3 \cdot 2^{10} - 3 = 3069$.
(d) 27
(e) 2520

5. (10 points) (a) How many bit strings contain exactly 4 zeros and 6 ones if every 0 is immediately followed by a 1?
(b) List the bit strings in part (a).

ANSWER:
(a) $C(6, 2) = 15$

6. (10 points) Prove the identity $C(2n, 2) = 2C(n, 2) + n^2$ where $n$ is a positive integer.

ANSWER:
Let $n$ be a positive integer. If $n \geq 2$ then $C(n, 2) = n!/(2!(n-2))!$ and we have

$$C(2n, 2) = \frac{(2n)!}{2!(2n-2)!} = \frac{(2n)(2n-1)}{2} = 2n^2 - n$$

and

$$2C(n, 2) + n^2 = 2 \frac{n!}{2!(n-2)!} + n^2 = n(n-1) + n^2 = 2n^2 - n.$$ 

Since the resulting are equal the identity is proved for $n \geq 2$. When $n = 1$ then $C(n, 2) = 0$ and both sides of the equation evaluate to 1.

7. (10 points) (a) Determine the coefficient of $x^9y^3$ in the expansion of $(x + y)^{12}$
(b) Determine the coefficient of $x^9y^3$ in the expansion of $(x - 3y)^{12}$
(c) Determine the coefficient of $x^0$ in the expansion of $(x + \frac{1}{x})^{12}$

ANSWER:
(a) 220  (b) $-5940$  (c) 220

8. (10 points) How many solutions to the equation $n_1 + n_2 + n_3 = 20$ are there if
(a) $n_1$, $n_2$ and $n_3$ are positive integers?
(b) $n_1$, $n_2$ and $n_3$ are integers larger than 5?

ANSWER:
(a) 171  (b) 6