Math 2513, Spring 2005
EXAM 1—Solutions

NOTE: Many of the problems on this test can be solved in more than one way.

1. Consider the set identity $A \subseteq A \cup B$. (a) Write down complete statements of the two definitions which play central roles in this identity. (b) Prove the identity.

(a) The two definitions which play a role in this statement are "subset" and "union", defined as follows: Let C and D be sets. We say C **is a subset of** D (and write $C \subseteq D$) provided that whenever x is an element of C then x is an element of D. (This can also be phrased as: if $x \in C$ then $x \in D$.) The **union of** A **and** B is the set $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

(b) Let A and B be sets. Suppose that x is an element of A. Then it is clearly true that "x is an element of A or x is an element of B". Thus $x \in A$ or $x \in B$, and this shows that $x \in A \cup B$ by the definition of union. So every element x of A is also an element of $A \cup B$. Using the definition of union, it follows that $A \subseteq A \cup B$. $\square$

2. Let $A = \{a, b, c, d, e, f\}$. (a) Let $\mathcal{P}(A)$ be the power set of A. How many elements does $\mathcal{P}(A)$ have? (b) List the elements of the subset of $\mathcal{P}(A)$ consisting of those subsets of A which have cardinality 4 and contain both c and d.

(a) The **power set of a set** A is the set $\mathcal{P}(A)$ of all subsets of A. When A is a finite set with n elements then it is known that $\mathcal{P}(A)$ is a finite set with 2^n elements. In this example the set $A = \{a, b, c, d, e, f\}$ has six elements so $\mathcal{P}(A)$ has $2^6 = 64$ elements. It would be tedious, but not difficult, to list out all 64 subsets. For example the respective numbers of subsets with 0, 1, 2, 3, 4, 5 and 6 elements are 1, 6, 15 20, 15, 6 and 1.

(b) To construct a four element subset of $A = \{a, b, c, d, e, f\}$ containing c and d, we choose exactly two elements from $\{a, b, e, f\}$ and then include c and d. There are six possibilities: $\{a, b, c, d\}$, $\{a, e, c, d\}$, $\{a, f, c, d\}$, $\{b, e, c, d\}$, $\{b, f, c, d\}$ and $\{e, f, c, d\}$.

3. Let B_8 be the set of all bit strings with length 8. Let $f : B_8 \to \mathbb{Z}$ be the function which assigns to each bit string α of length 8 the integer $n - m$ where n is the number of 1's in α and m is the number of 0's. (a) What are the domain, codomain and range of this function f? (b) Show (by example) that f is not injective. (c) Determine $f(S)$ if S is the subset of B_8 consisting of bit strings with an even number of 0's.

(a) From the given description we can immediately say that the domain of f is the set B_8 (consisting of all bit strings of length eight), and that the codomain of f is the set $\mathbb{Z}$ of integers. As is often the case, determining the range of f is a bit more difficult. Here observe that if α is a bit string with n ones then the number of zeroes in α is $m = 8 - n$, so $f(\alpha) = n - (8 - n) = 2n - 8$. Since n must equal one of the nine integers from 0 to 8 then we see that the range of f is the set $\{-8, -6, -4, -2, 0, 2, 4, 6, 8\}$ of even integers between -8 and 8 inclusive.

(b) Consider the bit strings $\alpha_1 = 11000000$ and $\alpha_2 = 00000011$ in B_8. Each of these bit strings has two 1's and six 0's, so $f(\alpha_1) = -4 = f(\alpha_2)$, and since $\alpha_1 \neq \alpha_2$ this shows that f is not an injective function.

(c) Let α be a bit string in S. The number of ones in α must equal 0, 2, 4, 6 or 8 (since if there are an even number of zeroes then there will also be an even number of 1's in α). Therefore, using the definition of $f(S)$ and the observations in part (a) gives

$$f(S) = \{f(x) \mid x \in S\} = \{2n - 8 \mid n \text{ equals } 0,\ 2,\ 4,\ 6 \text{ or } 8\} = \{-8, -4, 0, 4, 8\}.$$

4. Let $A = \{a, b, c\}$ and let $B = \{n \in \mathbb{Z} \mid n^2 + 2 < 6\}$.
(a) List the elements of the Cartesian products $A \times B$ and $B \times A$.
(b) What are the cardinalities of $A \times B$ and $(A \times B) \times (A \times B)$?

(a) Note that $B = \{n \in \mathbb{Z} \mid n^2 + 2 < 6\} = \{-1, 0, 1\}$ so

$$A \times B = \{(a, -1), (a, 0), (a, 1), (b, -1), (b, 0), (b, 1), (c, -1), (c, 0), (c, 1)\}$$

and

$$B \times A = \{(-1,a),(-1,b),(-1,c),(0,a),(0,b),(0,c),(1,a),(1,b),(1,c)\}$$

(b) $|A \times B| = |A| \cdot |B| = 3 \cdot 3 = 9$ and $|(A \times B) \times (A \times B)| = |(A \times B)| \cdot |(A \times B)| = 81$.

5. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ given by $f(x) = 3x - 2$ and $g(x) = 5x + 1$. (a) Give equations for $g \circ f$ and $f \circ g$ and show that the two functions are different. (b) Let $h : \mathbb{R} \to \mathbb{R}$ be a function for which $h \circ f(x) = x$ for every real number x. Find an equation for $h(x)$.

(a) We have

$$g \circ f(x) = g(f(x)) = g(3x - 2) = 5(3x - 2) + 1 = 15x - 9$$

and

$$f \circ g(x) = f(g(x)) = f(5x + 1) = 3(5x + 1) - 2 = 15x + 1 \ .$$

If we evaluate these functions at $x = 0$ we see that $g \circ f(0) = -9$ and $f \circ g(0) = 1$ which definitely shows that the functions are not equal.

(b) Let $h : \mathbb{R} \to \mathbb{R}$ be a function for which $h \circ f(x) = x$. Since $f(x) = 3x - 2$ this means that $h(3x - 2) = x$ for every real number x. Let's write $y = 3x - 2$. Then the equation $h(3x - 2) = x$ can be written as $h(y) = x$. Solving $y = 3x - 2$ for x gives $x = (y + 2)/3$, and therefore $h(y) = (y + 2)/3$. So the function h is described by the equation $h(x) = (x + 2)/3$.

6. Let A, B and C be sets, and let $f : A \to B$ and $g : B \to C$ be functions. Prove the statement: *If f and g are one-to-one then $g \circ f$ is one-to-one.*

proof: Let A, B and C be sets, and let $f : A \to B$ and $g : B \to C$ be functions. Assume that$f$ and g are one-to-one functions. Suppose that a_1 and a_2 are elements of A for which $g \circ f(a_1) = g \circ f(a_2)$. (To show that $g \circ f$ is one-to-one we must verify that $a_1 = a_2$.) Let $b_1 = f(a_1)$ and $b_2 = f(a_2)$. Then $g(b_1) = g(f(a_1)) = g \circ f(a_1) = g \circ f(a_2) = g(f(a_2)) = g(b_2)$. Since g is one-to-one and $g(b_1) = g(b_2)$ it follows that $b_1 = b_2$. Thus $f(a_1) = b_1 = b_2 = f(a_2)$, and so a_1 equals a_2 because f is one-to-one. We have shown that $a_1 = a_2$ whenever $g \circ f(a_1) = g \circ f(a_2)$, and so we conclude that $g \circ f$ is one-to-one. $\square$

7. Let A and B be sets such that $A \subseteq B$. Show that $A \cap B = A$.

proof: Let A and B be sets, and assume that $A \subseteq B$. Suppose that x is an element of $A \cap B$. By the definition of intersection, this means that x is element of both A and B, and, in particular, x is an element of A. Therefore every element x of $A \cap B$ is also an element of A. It follows that $A \cap B \subseteq B$ by the definition of subset.

Now suppose that y is an element of A. Since $A \subseteq B$, this implies that $y \in B$ by the definition of subset. Therefore we see that y is an element of both A and B, and so $y \in A \cap B$ by the definition of intersection. So we've shown that every element of A is an element of $A \cap B$, which means that A is a subset of $A \cap B$ by the definition of subset.

In the two previous paragraphs we showed that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$. Using the definition of set equality, it follows that $A \cap B = A$. $\square$

8. Let A, B and C be sets. Consider the set equation $\overline{A - B} \cap C = C - A$.
(a) Show that the equation is not true for all sets.
(b) Show that there are some sets for which the equation is true.
(c) Is there any subset relation between the sets $\overline{A - B} \cap C$ and $C - A$?

(a) As an example, choose $A = B = C = \{1, 2\}$. Then $\overline{A - B} \cap C = \{1, 2\}$ but $C - A = \emptyset$. This shows that the equation is not true for all sets A, B and C. (NOTE: More generally, if any three sets A, B and C are chosen where $A \cap B \cap C \neq \emptyset$ then the equation will fail.)

(b) Choose $A = B = C = \emptyset$. Then both sides of the equation are the empty set and the equation is true for that choice of sets.

(c) The subset relation $C - A \subseteq \overline{A - B} \cap C$ is valid. Its not hard to prove this formally, but a look at Venn diagrams will suggest its truth more quickly.