1. (15 points) Use induction to prove that \( \sum_{k=0}^{n} 2k + 1 = (n + 1)^2 \) for every \( n \in \mathbb{N} \).

2. (15 points) (a) Let \( A \) and \( B \) be sets. Give the definition of the set difference \( A - B \).
   (b) Use basic definitions to show that if \( A \) and \( B \) are disjoint sets then \( A - (A - B) = \emptyset \).

3. (5 points) Draw a schematic diagram of a directed graph which has (directed) paths of length 2 and 3 but no (directed) path of length 5.

4. (20 points) Let \( A \) be the set of all bit strings of length 10.
   (a) How many elements does \( A \) have?
   (b) Does \( A \) have more than \( 10^{10} \) subsets with exactly four elements?
   (c) Let \( C \) be the subset of \( A \) consisting of those 10-strings which contain a "100100" substring. How many elements does \( C \) have?
   (d) Let \( D \) be the subset of \( A \) consisting of those 10-strings which contain a "100100" substring or start with two successive 1's. Determine \( |D| \).
   (e) How many 10-strings contain a "100100" substring or a "00000" substring?

5. (20 points) (a) Let \( X \) and \( Y \) be sets and \( f : X \rightarrow Y \) be a function. Define what it means for \( f \) to be one-to-one.
   (b) Give an example of a function which is not one-to-one.
   (c) Give an example of a function which is one-to-one.
   (d) If \( X \) has \( n \) elements and \( Y \) has \( k \) elements, then how many one-to-one functions from \( X \) to \( Y \) are there?
   (e) Let \( X = \{x_1, x_2, x_3, x_4\} \) and \( Y = \{y_1, y_2, y_3, y_4, y_5\} \). How many one-to-one functions \( f : X \rightarrow Y \) are there that satisfy \( f(\{x_1, x_2\}) \subseteq \{y_1, y_2\} \)?

6. (15 points) Let \( r_1 \) and \( r_2 \) be rational numbers where \( r_2 \neq 0 \). Show that \( 2r_1 + \frac{r_1}{r_2} \) is rational.

7. (10 points) Let \( A = \{1, 2, 3\} \). (a) How many different relations are there on \( A \)? (b) Give an example of a relation on \( A \) which contains \((1, 3)\) but is neither symmetric nor anti-symmetric. (c) Give an example of a relation on \( A \) that is both symmetric and anti-symmetric.