Finding Points of Intersection of Two Curves given by parametric equations

Example: Let C_1 and C_2 be curves with

$$C_1 : \begin{cases} x = t^2+1 \\ y = t^3 - t \end{cases} \quad \text{and} \quad C_2 : \begin{cases} x = t+2 \\ y = -2t-2 \end{cases}$$

The first observation is that if P is a point of intersection of C_1 and C_2 then the time value when the first object is at P may likely be different than the time value for the second object. (If the two time values are the same then both objects are at P at the same instant, and we might say that P is a point where the two objects collide.)

To address this problem it is wise to use a different time variable for one of the curves. Here we'll express C_2 using the variable (or parameter) s:

$$C_2 : \begin{cases} x = s+2 \\ y = -2s-2 \end{cases}$$

So if the first object is at P at time t, and the second object is there at time s, we have two equations

$$\begin{cases} (1)\ t^2+1 = s+2 \\ (2)\ t^3 - t = -2s+2 \end{cases}$$

to solve (simultaneously) for s and t.

Solving equations like these can be algebraically difficult! But here we can proceed as follows.

$$① \Rightarrow \quad s = (t^2+1)-2 = t^2-1$$

and substituting this for s in ② gives

$$t^3 - t = -2(t^2-1)-2 = -2t^2$$

Do the algebra carefully!

which can be written as

$$0 = t^3+2t^2-t = t(t^2+2t-1)$$

So $t=0$, or, $t^2+2t-1=0$, and the second of these equations gives $t = \dfrac{-2 \pm \sqrt{(2)^2-4(1)(-1)}}{2} = -1 \pm \sqrt{2}$.

So it appears there are 3 points of intersection with these values of t, and we can go back to find $s = t^2-1$ and $(x,y) = (s+2, -2s-2)$.

This gives:

points of intersection

t	s	(x,y)
0	-1	$(1,0)$
$-1+\sqrt{2}$	$2(1-\sqrt{2})$	$(4-2\sqrt{2}, -6+4\sqrt{2}) \approx (1.17, -.34)$
$-1-\sqrt{2}$	$2(1+\sqrt{2})$	$(4+2\sqrt{2}, -6-4\sqrt{2}) \approx (6.83, -11.66)$

(and there are 3 distinct points of intersection.)