

Special Examples - Parametrizations

① Circles. For constants $R > 0$, x_0, y_0 the parametric equations

$$C: \begin{cases} x = R \cos(t) + x_0 \\ y = R \sin(t) + y_0 \end{cases}$$

describes the circle of radius R centered at (x_0, y_0) which has rectangular equation $(x - x_0)^2 + (y - y_0)^2 = R^2$. (Check this by substituting $x = R \cos(t) + x_0$ and $y = R \sin(t) + y_0$ into the rectangular equation.)

The motion described by the parametric equations goes counterclockwise and goes once around the circle in each time interval of length 2π .

The speed of the object at time t is

$$s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(-R \sin t)^2 + (R \cos t)^2} = \sqrt{R^2} = R$$

which is constant.

To traverse the circle clockwise, you could replace t by $-t$ in the above getting

$$\begin{cases} x = R \cos(t) + x_0 \\ y = -R \sin(t) + y_0 \end{cases} \quad \begin{array}{l} \text{because } \cos(-t) = \cos(t) \\ \text{and } \sin(-t) = -\sin(t) \end{array}$$

Note Well! This is only one of many, many other parametrizations of the circle $(x - x_0)^2 + (y - y_0)^2 = R^2$. But if you have to choose one of these parametrizations to use, this is probably the one you would want.

Example This parametrization suggests how to work with ellipses also.

For instance, the ellipse $25x^2 + 4y^2 = 100$ has parametric

equations $\begin{cases} x = 2 \cos t \\ y = 5 \sin t \end{cases} \quad 0 \leq t \leq 2\pi$

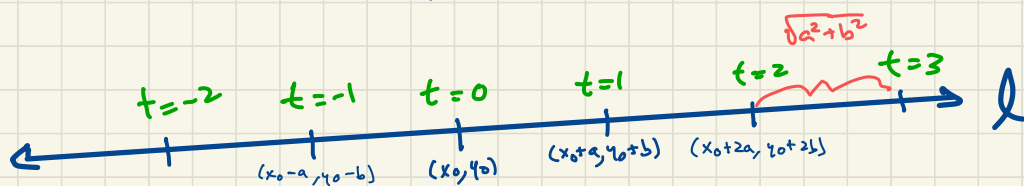
② Lines. If $a \neq 0, b, x_0, y_0$ are constants then

$$l: \begin{cases} x = at + x_0 \\ y = bt + y_0 \end{cases}$$

describes a nonvertical line l in \mathbb{R}^2 . This line goes thru (x_0, y_0) (when $t=0$) and (x_0+a, y_0+b) (when $t=1$) and has slope b/a .

A rectangular equation is $y - y_0 = \frac{b}{a}(x - x_0)$, which can also be written with the form $y = \frac{b}{a}x + \frac{ay_0 - bx_0}{a}$ or by $ay - bx = ay_0 - bx_0$.

The speed of this motion is $s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2 + b^2}$ so the object is moving with constant speed.



So this parametrization amounts to identifying l as a number line in which the unit length is $\sqrt{a^2 + b^2}$.

Note Well! The same line l can be identified with many, many different number lines by rechoosing the unit length and/or rechoosing the "origin" (point where $t=0$). So one line l has many different parametrizations of the form described here. On top of that there are many parametric descriptions of l that don't have constant speed.

In section 12.5, we will ^{examine} these parametrizations of lines more thoroughly.

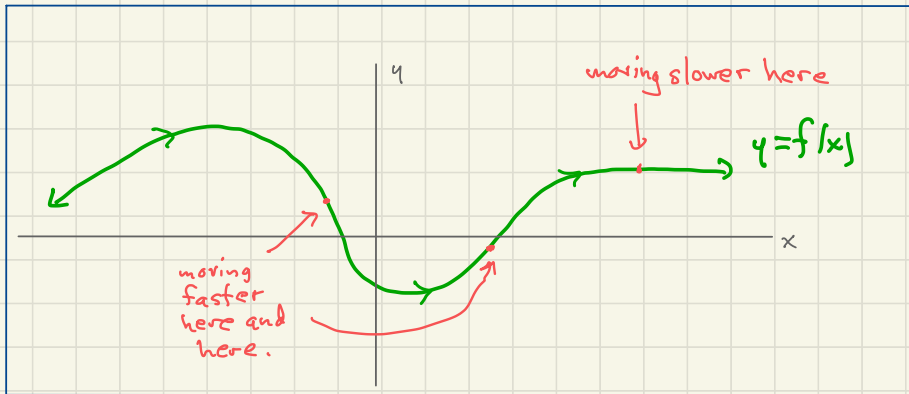
③ Graphs of Functions.

The graph $y=f(x)$ of a function $f(x)$ can be parametrized by:

$$C: \begin{cases} x=t \\ y=f(t) \end{cases}$$

Since $\frac{dx}{dt}=1$, the object described by the equations has constant horizontal speed from left to right. However the actual speed at time t is

$$s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1 + f'(t)^2}$$



Note Well! This is only one of many, many other parametrizations of $y=f(x)$ (a curve that satisfies VLP). But if you have to choose one of these parametrizations to use, this is probably the one you would want.

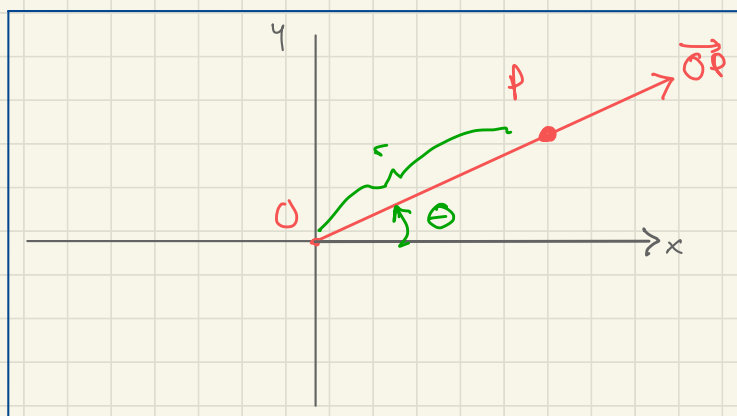
Polar Coordinates

Any point P in the xy -plane can be located by specifying :

(1) Its distance $r = \text{dist}(O, P)$ from the origin , and

(2) The angle θ from the positive x -axis to the ray \overrightarrow{OP} .

We say that the ordered pair (r, θ) are polar coordinates for P .



With polar coordinates there are some "non-uniqueness" issues to be aware of:

(i) The angle θ is only unique up to integer multiples of 2π .

- e.g. $P = (r, \theta) = (r, \theta + 2\pi) = (r, \theta - 2\pi) = (r, \theta + 6\pi)$, etc.

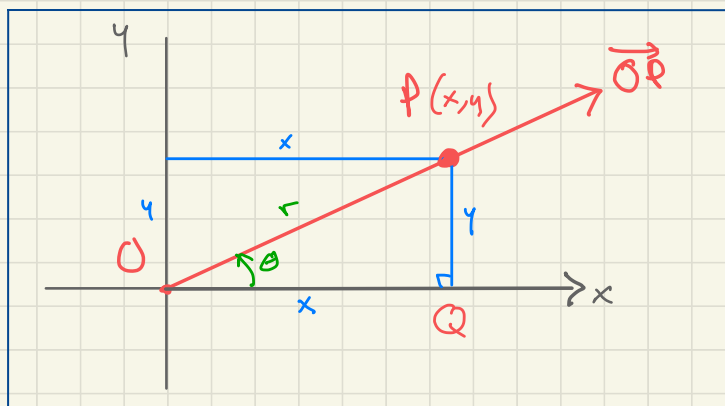
(ii) We sometimes interpret r as "directed distance" meaning that it could be negative. (In this case think of moving "backwards" along \overrightarrow{OP} to get to (r, θ) .)

- $(-r, \theta) = (r, \theta + \pi)$

(iii) When P is the origin, the angle θ doesn't make sense

- That is, the origin O equals $(0, \theta)$ for any value of θ .

Relationship between rectangular and polar coordinates:



From the right triangle $\triangle OQP$ we see that
 $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$ and $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$ which shows that

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

So these equations show how to determine the rectangular coordinates of a point from its polar coordinates.

Example The point with polar coordinates $(3, \pi/3)$ has rectangular coordinates $(3 \cos \pi/3, 3 \sin \pi/3) = (\frac{3}{2}, \frac{3\sqrt{3}}{2})$.

Example The point with polar coordinates $(3, 10)$ lies in the third quadrant because $x = 3 \cos(10)$ and $y = 3 \sin(10)$ are both negative. You could also see this by observing that $3\pi < 10 < 7\pi/2$.