Example We will examine the curve $C: \begin{cases} x = t^3 - 3t \\ y = t^3 - 3t^2 \end{cases}$

① Find an equation for the line tangent to C at the point where $t = -1/2$.

- $t = -1/2 \Rightarrow (x,y) = \left((-1/2)^3 - 3(-\frac{1}{2}),\ (-\frac{1}{2})^3 - 3(-\frac{1}{2}) \right) = (11/8, -7/8)$
- Slope of tangent line $= \left.\frac{dy}{dx}\right|_{t=-1/2}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 6t}{3t^2 - 3} = \frac{t^2 - 2t}{t^2 - 1}$$

$$\left.\frac{dy}{dx}\right|_{t=-1/2} = \frac{(-1/2)^2 - 2(-1/2)}{(-\frac{1}{2})^2 - 1} = -\frac{5}{3}$$

- Desired Equation: $y + 7/8 = -\frac{5}{3}\left(x - \frac{11}{8}\right)$

  , or, $y = -\frac{5}{3}x + \frac{17}{12}$ (slope-intercept form)

② At the point where $t = -1/2$ is C concave up, or concave down?

We need to calculate $\left.\frac{d^2y}{dx^2}\right|_{t=-1/2}$ and see if it is positive or negative.

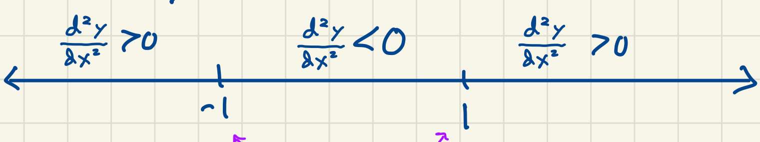
- $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{1}{dx/dt} \cdot \frac{d}{dt}\left[\frac{t^2-2t}{t^2-1}\right]$ (use Quotient Rule)

$$= \frac{1}{3t^2-3} \cdot \frac{(2t-2)(t^2-1) - (t^2-2t)(2t)}{(t^2-1)^2} = \frac{2}{3} \cdot \frac{t^2-t+1}{(t^2-1)^3}$$

- $\left.\frac{d^2y}{dx^2}\right|_{t=-\frac{1}{2}} = -\frac{224}{81} < 0 \Rightarrow$ concave down at that point

③ Any points of inflection? $\frac{d^2y}{dx^2} = \frac{2}{3}\frac{t^2-t+1}{(t^2-1)^3}$

This never equals 0 but it is undefined when $t = -1$ or $t = 1$



Points of Inflection

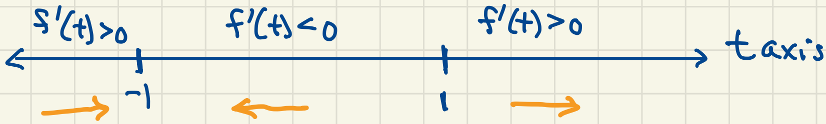
example, continued ...

④ Determine information to sketch the curve C.

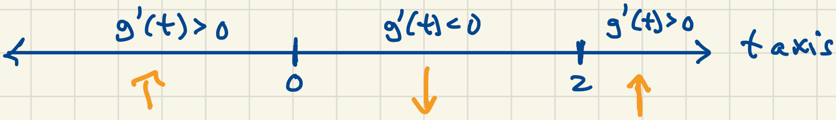
Basic principles for $C: x = f(t), \ y = g(t)$

- $f'(t) > 0$ when object moving right, $f'(t) < 0$ when object moving left
- $g'(t) > 0$ when object moving up, $g'(t) < 0$ when object moving down.

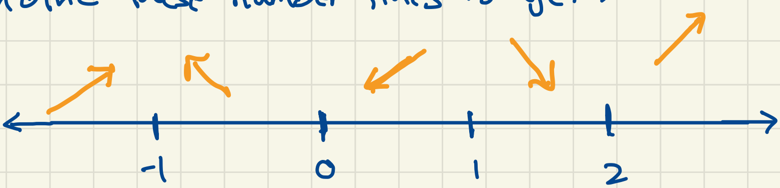
$$f'(t) = \frac{dx}{dt} = 3t^2 - 3 = 3(t-1)(t+1) \Rightarrow \text{critical numbers at } t=-1,\ t=1$$



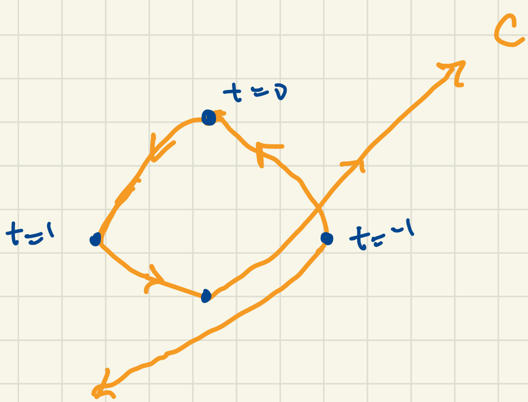
$$g'(t) = \frac{dy}{dx} = 3t^2 - 6t = 3t(t-2) \Rightarrow \text{critical numbers at } t=0,\ t=2$$



Combine these number lines to get:



⇒ rough shape of C is :



⑤ Now identify key points

t	(x,y)
-1	$(2,-4)$
0	$(0,0)$
1	$(-2,-2)$
2	$(2,-4)$
3	$(18,0)$
$-\sqrt{3}$	$(0, 3\sqrt{3}-9)$
$\sqrt{3}$	$(0, -3\sqrt{3}-9)$

Note object is at $(2,-4)$ at time $t=-1$ and again at time $t=2$.

This is a point of self-intersection on C !

Comment: To determine if $C. \begin{cases} x=f(t) \\ y=g(t) \end{cases}$ has any points of self-intersection algebraically, one needs to find distinct numbers t and s satisfying the simultaneous equations:

$$\begin{cases} f(t)=f(s) \\ g(t)=g(s) \end{cases}$$

That's generally a difficult algebra probem to solve.

$$C: \begin{cases} x = t^3 - 3t \\ y = t^3 - 3t^2 \end{cases}$$

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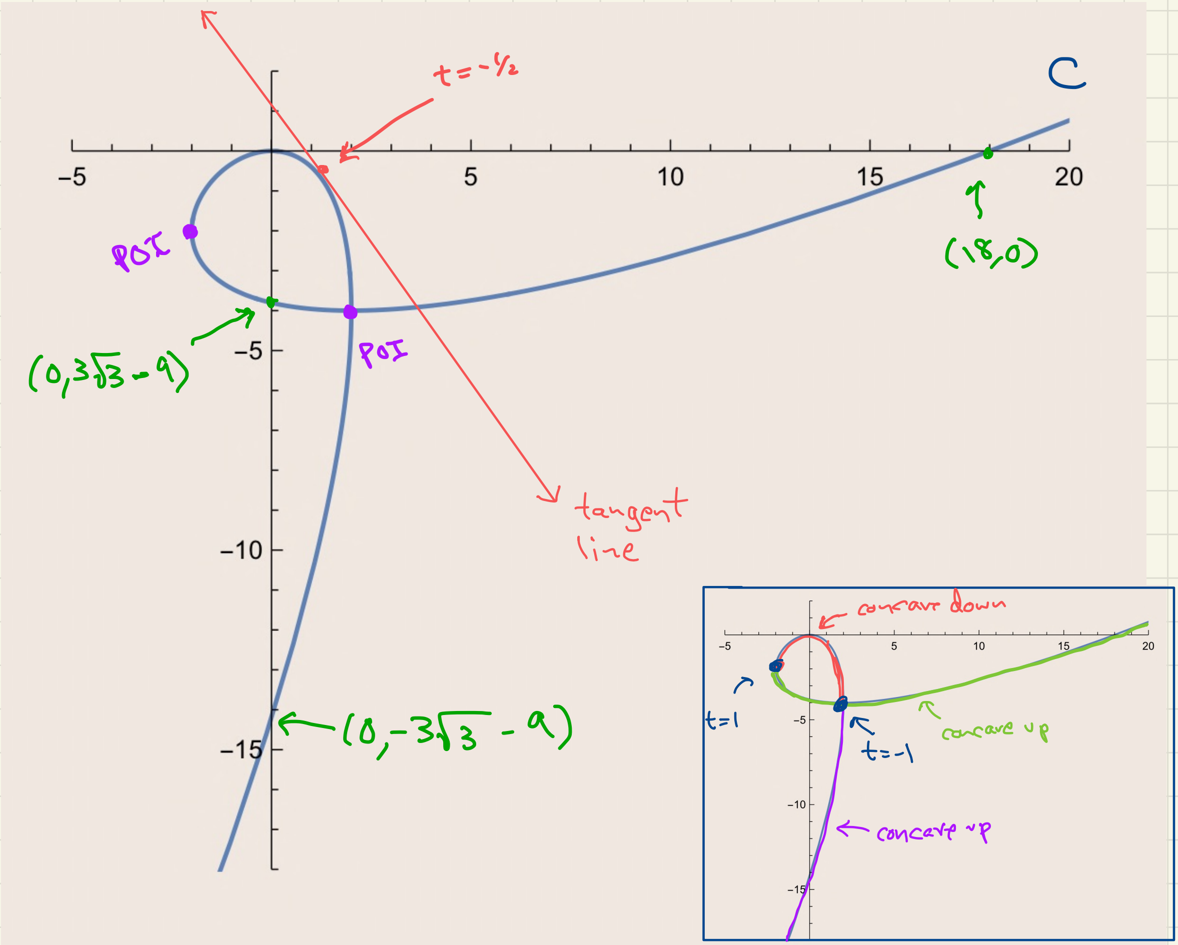

NOTE: This curve C has 2 x-intercepts, 3 y-intercepts

x-intercepts ($y=0$): $(0,0)$, $(18,0) \leftarrow t=3$

y-intercepts ($x=0$): $(0,0)$, $(0, 3\sqrt{3}-9)$, $(0, -3\sqrt{3}-9)$

$t=\sqrt{3}$ $\qquad$ $t=-\sqrt{3}$

$$C: \begin{cases} x = t^3 - 3t & = f(t) \\ y = t^3 - 3t^2 & = g(t) \end{cases}$$

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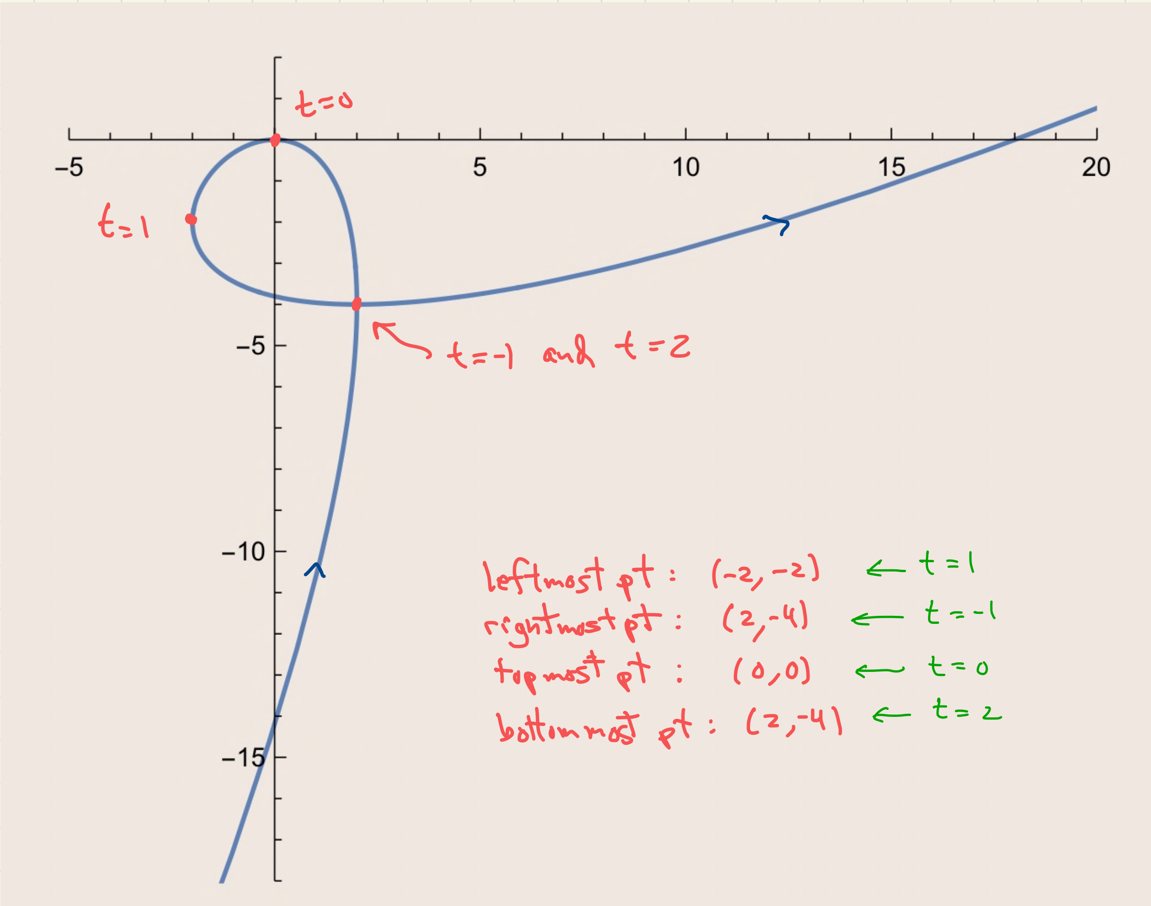


leftmost point = local min for $f(t)$
rightmost point = local max for $f(t)$

topmost point = local max for $g(t)$
bottommost point = local min for $g(t)$

If an object in motion is described by equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

then the speed of the object at time t is $\sqrt{f'(t)^2 + g'(t)^2}$.

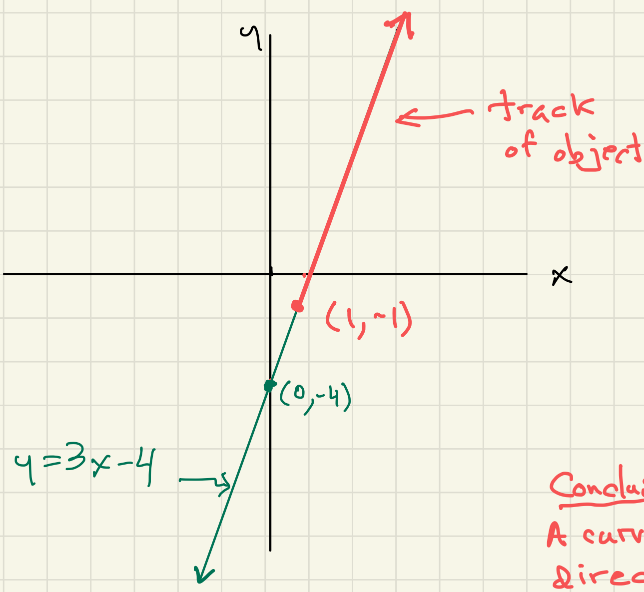
Example $\quad x = t^2+1,\ y = 3t^2-1$

speed at time $t = \sqrt{(2t)^2 + (6t)^2} = 2\sqrt{10}\,\sqrt{t^2} = 2\sqrt{10}\,|t|$.

To describe the motion in this example we could eliminate the parameter:

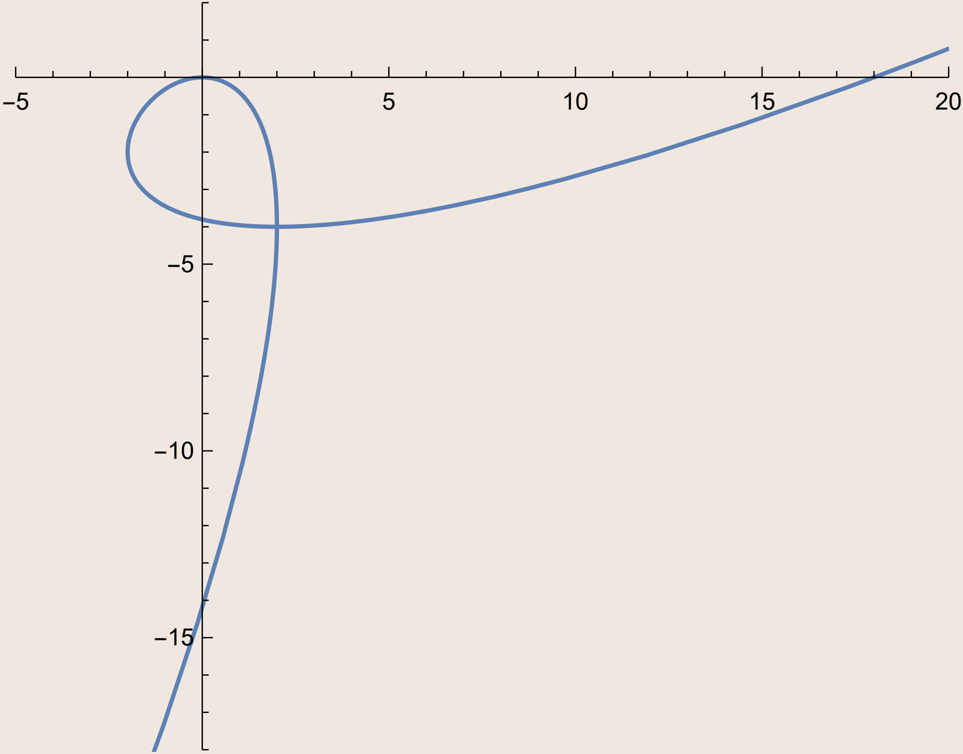
$$t^2 = x-1 \implies y = 3(x-1)-1 = 3x-4$$

So the motion takes place along the line $y=3x-4$. However the object does not trace the entire line because $x = t^2+1 \geq 1$. So it traces the portion of the line on or to the right of the line $x=1$.



So the object moves along this red ray to left, gradually slowing down, till it comes to a stop at $(1,-1)$. Then it moves back to the right, picking up speed as it moves farther from the origin.

Conclusion:
A curve can have a sudden change of direction at points where the speed is 0 (object comes to a stop).



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