Strategy for finding scalar equations for a line:

① Find a point $P = (x_0, y_0, z_0)$ on ℓ.

② Find a vector $\vec{d}_\ell = \langle a, b, c \rangle$ parallel to ℓ ("direction vector")

Then equations for ℓ are:

$$\ell : \begin{cases} x = at + x_0 \\ y = bt + y_0 \\ z = ct + z_0 \end{cases}$$

Strategy for finding the equation of a plane $\mathcal{P}$

① Find a point $P = (x_0, y_0, z_0)$ on $\mathcal{P}$.

② Find a vector $\vec{N}_{\mathcal{P}} = \langle a, b, c \rangle$ perpendicular to $\mathcal{P}$ ("normal vector").

Then an equation for the plane is

$$\mathcal{P} : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

A curve C in xyz-space can be described by parametric equations:

$$C: \begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases} \qquad a \leq t \leq b$$

may or may not have a restriction like this.

Or express these equations in vector form:

$$C: \vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad , \quad a \leq t \leq b$$

we say $\vec{r}(t)$ is a vector function

We can then differentiate:

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$
$$= \lim_{h \to 0} \frac{1}{h} \left(\vec{r}(t_0 + h) - \vec{r}(t_0) \right).$$

Fact If we think of $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ describing the motion of an object along a curve C, then the velocity at time $t = t_0$ is $\vec{r}\,'(t_0) = \vec{v}(t)$, and this is a vector tangent to C at the point where $t = t_0$.

Also, the speed of object at time t_0 is

$$s(t) = |\vec{r}\,'(t_0)| = \sqrt{x'(t_0)^2 + y'(t_0)^2 + z'(t_0)^2}$$

The acceleration at time t_0 is

$$\vec{a}(t) = \vec{r}\,''(t_0) = \langle x''(t_0), y''(t_0), z''(t_0) \rangle$$

Example An object moves along a curve C in 3-space according to the vector function

$$C: \vec{r}(t) = \langle 3t^2/2,\ 5t+1,\ 2t^2-t \rangle$$

ⓐ Show that the point $P = (6, -9, 10)$ is on C.

ⓑ Find equations for the line ℓ tangent to C at P.

ⓒ A plane p intersects C perpendicularly at P. Find an equation for p.

Example (continued)

$$C: \vec{r}(t) = \langle 3t^2/2,\ 5t+1,\ 2t^2 - t \rangle$$

At what time (if any) is the speed a minimum?

$$\vec{r}\,'(t) = \langle 3t, 5, 4t-1 \rangle$$

$$s(t) = |\vec{r}\,'(t)| = \sqrt{(3t)^2 + (5)^2 + (4t-1)^2}$$

$$= \sqrt{9t^2 + 25 + 16t^2 - 8t + 1}$$

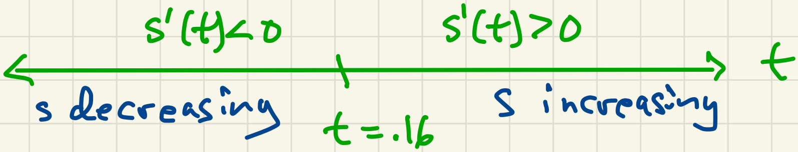
$$= \sqrt{25t^2 - 8t + 26}$$

Useful simplification: local extremes for $s(t)$ occur at same values of t for local extremes of

$$s(t)^2 = 25t^2 - 8t + 26$$

$$\frac{d}{dt}\left[25t^2 - 8t + 26\right] = 50t - 8$$

critical points $50t - 8 = 0 \Rightarrow t = 8/50 = .16$



Answer Speed has a absolute minimum value when $t = .16$ and object is located at $(.0384, 1.8, -1.9488)$

Example $C: \vec{r}(t) = \langle \cos t, \sin t, 3t \rangle$

Find $\vec{v}(t)$, $s(t)$, $\vec{a}(t)$:

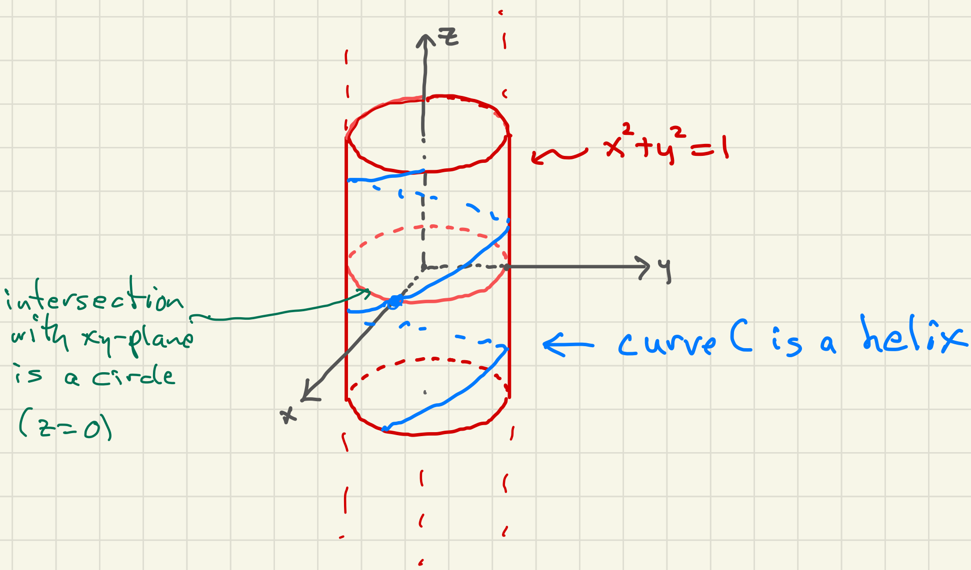
$$\vec{v}(t) = \vec{r}\,'(t) = \langle -\sin t, \cos t, 3 \rangle$$

$$s(t) = \sqrt{(-\sin t)^2 + (\cos t)^2 + 3^2} = \sqrt{10}$$

$$\vec{a}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

The motion described by $\vec{r}(t)$ has constant speed but neither velocity nor acceleration is constant.

Note: Each point $(\cos t, \sin t, 3t)$ on C satisfies the equation $x^2+y^2=1$. The set of all points satisfying $x^2+y^2=1$ is a cylinder in 3-space whose axis is the z-axis.



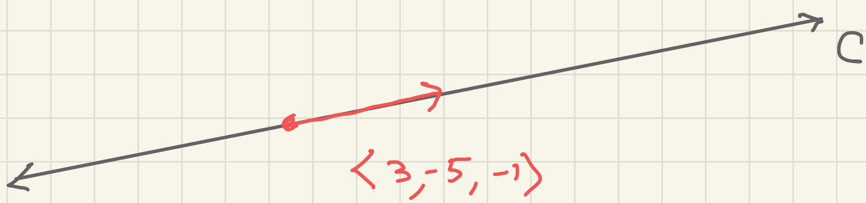
Example Find velocity, speed, acceleration for the vector function: $\vec{r}(t) = \langle 3t-2, -5t+1, -t+7 \rangle$

$$\vec{r}\,'(t) = \langle 3, -5, -1 \rangle \quad \leftarrow \text{constant vector}$$

$$s(t) = |\langle 3, -5, -1 \rangle| = \sqrt{35} \quad \leftarrow \text{constant speed}$$

$$\vec{a}(t) = \vec{r}\,''(t) = \langle 0, 0, 0 \rangle = \vec{0}$$

discussion: The curve described by $\vec{r}(t)$ here is a line with direction vector $\vec{d} = \langle 3, -5, -1 \rangle$. The object is moving with constant speed along this line.



Any object in motion having constant velocity will trace out a line in 3-space, where the object is moving along the line with constant speed.

Example The curve associated with the vector function

$$\vec{r}(t) = t\vec{d} + \vec{r}_0 = t\langle a,b,c\rangle + \langle x_0,y_0,z_0\rangle$$

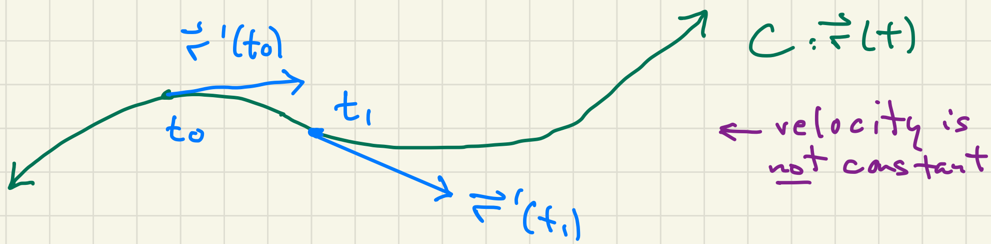
is a straight line and

$$\vec{r}\,'(t) = \vec{d} = \langle a,b,c\rangle = \vec{v}(t)$$

So these equations represent the motion of an object in 3-space with constant velocity.
Also $s(t) = \sqrt{a^2+b^2+c^2}$ is constant and $\vec{a}(t) = \vec{0}$.

Note: Having constant velocity is a much stronger condition than having constant speed.

Having constant velocity guarantees the object is moving along a straight line.



Fact $\vec{r}\,'(t_0) = \vec{v}(t_0)$ is a vector tangent to C at point on C where $t = t_0$.