Which can be solved using ratio test?

11.7 EXERCISES

1–38 Test the series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$

2. $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1}$

4. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$

5. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

6. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1+n)^{3n}}$

7. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$

9. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$

12. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

15. $\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$

16. $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$

17. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

20. $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$

21. $\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$

22. $\sum_{k=1}^{\infty} \frac{1}{2+\sin k}$

23. $\sum_{n=1}^{\infty} \tan(1/n)$

24. $\sum_{n=1}^{\infty} n \sin(1/n)$

25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

27. $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

32. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

33. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

34. $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$

35. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

36. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

37. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$

38. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$

- The ratio test would be the first choice for:

  # 8, 9, 13, 17, 25

- It could also be used with for:

  # 5, 6, 10, 14, 15, 32, 37

(But in these problems other tests are easier.)

- It would be inconclusive (that is, useless) for determining convergence in the remaining problems.

A power series centered at a has form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

It defines a function $f(x)$ whose domain is an interval I with endpoints $a-R$ and $a+R$, where $R \geq 0$ is the radius of convergence of the power series.

example $\quad f(x) = \sum_{n=0}^{\infty} n!\, x^n$

$(n+1)! = (n+1) \cdot n!$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!\, |x|^{n+1}}{n!\, |x|^n} = (n+1)\,|x| \xrightarrow[n\to\infty]{} \infty \ (\text{when } x \neq 0)$$

$\infty = L$

since $\infty \not< 1$, the series diverges for every $x \neq 0$.

$\Rightarrow \quad R = 0, \quad I = \{0\}$

example $\quad f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad = \quad$ A familiar function ?

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0 = L$$

so series converges for all $x \Rightarrow R = \infty$, $I = (-\infty, \infty) .= \mathbb{R}$

$$\boxed{\begin{array}{l}\text{Question: Which functions } f(x) \text{ can be expressed as} \\ f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{where } R > 0 \text{ ?}\end{array}}$$

Some examples   All of following have $R=1$, $a=0$.

① $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ Geometric Series

② $f(x) = \frac{x^4}{1-x} = x^4 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+4} = \sum_{n=4}^{\infty} x^n$

③ $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \overset{\text{use ①}}{=} \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

$= 1 - x^2 + x^4 - x^6 + \ldots$

④ $\int \frac{1}{1+x^2}\,dx \underset{*}{=} C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

$\Rightarrow \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$

* This uses "term-by-term integration".

note: In examples ①, ②, ③, $I = (-1,1)$.
But in ④, $I = (-1,1]$.

So putting $x=1$ in ④ gives

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

→ use this to approximate π.

Theorem If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$ then $f'(x)$ and $\int f(x)\,dx$ can be expressed as power series with same radius of convergence R:

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_2(x-a)^2 + \cdots$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$
$$= C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \cdots$$

(called term-by-term differentiation and integration.)

Example $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (revisited) $R = \infty$

$$= \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots$$

$$\Rightarrow \quad f'(x) = 0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$= f(x)$$

$\Rightarrow f(x) = Ce^x$ and $1 = f(0) = Ce^0 = C$

Conclude: $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + \cdots$$

$$f'''(x) = 6c_3 + 24(x-a) + \cdots$$

$$f^{(n)}(x) = n!\,c_n + (\text{Stuff})\cdot(x-a)$$

Now plug $x=a$ into these

$$f(a) = c_0$$
$$f'(a) = c_1$$
$$f''(a) = 2c_2$$
$$f'''(a) = 6c_3$$
$$f^{(n)}(a) = n!\,c_n$$

$\Leftarrow$ Shows that $c_n = \dfrac{f^{(n)}(a)}{n!}$

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ with $R>0$, then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$ is called the "Taylor series for $f(x)$ at $x=a$".

$\boxed{\text{If } f(x) \text{ equals a power series with } R>0 \text{ then that power series must be the Taylor series.}}$

example what is Taylor series for $f(x)=\sin(x)$ for $a=0$?

Answer $$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$